Chapter 3
The Role of Algebra and Early Algebraic Reasoning in the Australian Curriculum: Mathematics

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In this chapter we describe the place of algebra (Number and Algebra [NA] content strand) and the development of early algebraic reasoning in *The Australian Curriculum: Mathematics*). We raise critical questions about the scope, depth and sequence of the algebra (NA) strand and the interrelationships between the Proficiencies and Number and Algebra. This is informed by a review of the research bases for prioritising algebra within mathematics curriculum reform generally. We describe the potential affordances and limitations of algebra for mathematics teaching and learning from preschool through secondary grades. The challenges facing teachers and the impact on the development of teachers' pedagogical content knowledge are discussed. Implications for further research on the implementation of the algebra strand within the Australian context are outlined.

Introduction

Algebra is considered an integral part of mathematics curriculum. In recent decades the view that algebra should be introduced in the primary grades has been more widely accepted with even more recent emphasis on patterning and early algebraic reasoning in the preschool and early grades. Early algebraic thinking develops through an awareness of the structural relationships of patterns and later in the structure of arithmetic (Carraher, Schliemann, Brizuela & Earnest, 2006; Kaput, 2008; Mason, Stephens, & Watson, 2009). Patterning in the early years often does not involve numbers but can still lead to simple forms of generalisation. Mason, Graham, and Johnston-Wilder (2005) argue for a focus on generalisation in early mathematics learning. Early algebraic reasoning can develop from a natural awareness of generalisation and ability to express generality. Thus the role of algebraic thinking in the curriculum is about promoting generalisation in mathematics and reflecting this in a scope and sequence throughout the primary

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and secondary years. Moreover Warren asserts that early algebraic thinking refers not only to thinking about algebra early, but is also about re-looking at number from a more structural perspective (Warren, personal communication, September, 2011).

Research Bases for Prioritising Algebra within Mathematics Curriculum Reform

In past decades research on the teaching and learning of algebra has largely focused attention on secondary students' difficulties. However, the problems associated with the teaching and learning of algebra can be traced to the development, or lack thereof, of early algebraic reasoning right back to the prior-to-school context. Increasing interest in research about algebra in the primary school has ensued in order to investigate students' understanding of arithmetic based on mathematical generalisations and understanding of basic algebraic principles (Davis, 1985). More broadly, there is now a growing body of research on the development of early algebraic reasoning in the pre-primary school years (Blanton & Kaput, 2005; Carraher & Schliemann, 2007; Warren & Cooper, 2008; Yeap & Kaur, 2008) and prior to formal schooling (Papic, Mulligan & Mitchelmore, 2011).

There is also recognition in the research that arithmetic and algebraic thinking can and should be intertwined in the early years with each supporting the other. Thus it is crucial that algebraic thinking is developed in parallel with arithmetic thinking from an early age. Not only does this result in a deeper understanding of our number system but also forms a strong basis on which to build formal algebraic thinking in the later years. Generalisations from patterning involve inductive reasoning which research has shown that young students are capable of. This has implications for the curriculum and the way we teach arithmetic in primary contexts. One is that patterning and generalisations can only be reached from a range of examples in both numerical and non-numerical contexts. Thus the focus moves away from answering particular problems to exploring an array of examples of that particular problem type – with a focus on the structural aspects (Warren, personal communication, September, 2011).

Learning frameworks and programs and aligned professional development initiatives are now focusing on patterning and structural relationships in mathematics, including equivalence, growing patterns and functional thinking (Mulligan & Mitchelmore, 2009; Mulligan, English, Mitchelmore, Welsby & Crevensten, 2011; Warren & Cooper, 2008). These initiatives integrate important aspects of counting and arithmetic, common to numeracy programs but focus attention on common underlying processes that develop mathematical generalisation.

Current studies are providing increasing evidence that early algebraic thinking develops from the ability to see and represent patterns and relationships. Papic et al. (2011) found that preschoolers' ability to recognise the structure of a simple pattern is central to the notion of unit of repeat and the development of composite units necessary for understanding multiplication. Other research shows strong interrelationships between spatial structuring, patterning and number concepts (van Nes & de Lange, 2007) and the relationship between analogical reasoning and patterning (English, 2004).

New research and cross-cultural comparisons of mathematics curricula and achievement indicate that children in the 4–8 year age range can develop many concepts and processes described as typical of students from 8–12 years of age (i.e. equivalence, functions, complex repetition, multiplicative thinking and proportional reasoning). There is strong current and past research evidence that children in the K–2 years and prior to formal schooling are able to use models, pictures, and symbols to represent ideas, and justify and form simple generalisations. Thus, a new Australian curriculum presents opportunities to develop early algebraic reasoning through the Number and Algebra strand based on growing research evidence with younger students that was not available even a decade ago.

The Role of Algebra in Mathematics Curriculum

Over two decades ago, attempts to provide coherent goals and content for Australian mathematics curriculum resulted in *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990). A content strand was exclusively devoted to algebra and, in turn, influenced the development of algebra in state-based curriculum reviews. Bands A/B descriptions highlighted the importance of developing algebraic thinking during the primary years and asserted that the notions of generality, variation and function and unknown quantity are critical, not only to algebra but are implicit in the content of other strands of mathematics.

Most state-based Australian mathematics syllabi K-6 implemented in the past decade or more have included algebra as a strand of mathematical content or as central to working mathematically or problem solving. Patterning has been often paired as 'patterns and algebra' and usually linked to the number strand, and perhaps chance and data exploration. For example, in the current NSW Mathematics K-6 syllabus (Board of Studies NSW, 2002) the patterns and algebra strand is linked to the structure of arithmetic with the intent that students will develop knowledge, skills and understanding in patterning, generalisation and algebraic reasoning. Similarly in the Victorian mathematics curriculum algebra forms a central thread of the "structure" strand. The idea that algebraic reasoning and generalisation can be

developed throughout the K-6 curriculum has been espoused but key ideas regarding structural relationships such as equivalence and function have not been developed from Kindergarten/Prep as a coherent sequence. Early algebraic reasoning, with the intent to encourage generalisation through mathematical patterns, relationships and the structure of arithmetic is not really expected until late in the primary and early secondary grades.

Another issue is that even though some aspects of algebra may be visible, perhaps the necessary pedagogical content knowledge is not well formed by many teachers and thus implementation has been limited (Asquith, Stephens, Knuth, & Alibali, 2007; Ball, Thames & Phelps, 2008).

The Place of Algebra (Number and Algebra [NA] Content Strand) in the Australian Curriculum: Mathematics

The *Shape of the Australian Curriculum: Mathematics* paper (Australian Curriculum, Assessment and Reporting Authority, 2009) views the new curriculum as an opportunity for all students to access the study of mathematics. "The study of algebra clearly lays the foundations not only for specialised mathematics study but for vocational aspects of numeracy" (p. 11). Further, participation in algebra, for example, is connected to finishing high school and participation in the workforce. However, students' difficulties in studying algebra during the compulsory years often contribute to poor achievement and low retention rates in mathematics.

The Scope, Depth and Sequence of the Patterns and Algebra Sub-strand

Algebra is combined with number as the Number and Algebra strand, with the algebra content organised into two sub-strands: Patterns and Algebra (Foundation to Year 10) and Linear and Non-linear Relationships (Year 7 to Year 10).

Number and Algebra content strands are paired, as each enriches the study of the other. The underlying principle is that generalising arithmetic operations needs to take place in the primary school years in order to foster a deeper understanding of number and number operations, and to provide a bridge to the more formal study of algebra in the junior high school (Jacobs, Franke, Carpenter, Levi, & Battey (2007). Research in number properties and operations articulates the important connection between general structural principles underlying numerical relationships (Mason et al., 2009). For example, $a - b + b = a$ where the numbers operate as quasi-variables since the numerical relationships hold for all values and can be generalised (Fujii & Stephens, 2008). Missing number sentences are another example of structural numerical relationships that can be expressed algebraically, for example, $\Delta + 23 = 19 + 34$ or $26 \times 3.5 = \Delta \times 10$. "A deep understanding of equivalence and compensation is at the heart of structural thinking and arithmetic"

(Mason et al., 2009, p. 23). This provides a foundation for understanding equivalence relationships that underpin algebraic thinking. For younger children the same notion is evident in equivalent stories (Warren & Cooper, 2009) or simple arithmetic word problems where the structure of the problem may be a 'missing addend' or 'start unknown' situations. Blanton and Kaput refer to 'algebrafying arithmetic' where generalising arithmetic structures is a way of developing algebraic reasoning (Blanton & Kaput, 2003). Viewing simple functional relationships using a table of values for example, allows the numerical relationships to be expressed as pattern and as a generality whether it is symbolised algebraically or not.

> Students apply number sense and strategies for counting and representing numbers. They explore the magnitude and properties of numbers. They apply a range of strategies for computation and understand the connections between operations. They recognise patterns and understand the concepts of variable and function. They build on their understanding of the number system to describe relationships and formulate generalisations. They recognise equivalence and solve equations and inequalities. They apply their number and algebra skills to conduct investigations, solve problems and communicate their reasoning (ACARA, 2011, p. 2).

Although the aim is to build strong interrelationships between Number and Algebra, and the curriculum does provide a more integrated treatment of number and algebra, the content elaborations do not sufficiently emphasise the structural relationships. For example, the sub-strand Patterns and Algebra is really limited in the early years to natural patterns, simple repetition with objects and numbers, and skip counting. This restricts the curriculum from the very foundations to a traditional concept of pattern as simple repetition, rather than providing a more coherent overview of the important interrelationships between number and algebra, such as equivalence. In the secondary years (7 -10) although the concept of a variable is covered is some depth in Year 7 the focus shifts to simplifying and manipulating symbolic expressions such as procedures for factorisation and expansion. Explicit links back to number properties as the basis for developing algebraic relationships are not provided as guidance for teachers.

> A more integrated treatment of number and algebra, rather than just 'pairing' the sub-strands, provides opportunities for teachers to plan learning in a connected way. This allows students to understand more deeply number and number operations, especially in the middle and upper primary years. Similarly a more unified treatment of number and algebra can assist students to make a smoother transition to formal algebra in the secondary school (Stephens, personal communication, Sept, 2011).

Number and algebra might have also been paired with measurement or statistics and probability. The broader view of algebraic thinking might have also been linked to spatial aspects or graphical representations. Further, the interrelationship

between concepts proposed in the rationale seems to be contradicted in the content strand diagram that bears no relationship to interconnectedness between, for example, statistics, number, and algebra. Further, the diagram appears to indicate that there is a hierarchy of concepts beginning with number and algebra, and becoming increasingly layered to include more sophisticated concepts. The explicit role of algebraic reasoning in the Proficiencies strand in the context of problem solving is difficult to envisage.

Patterns and Algebra from Foundation to Year 2

Although the intent of the Foundation to Year 2 section is to describe the basic foundations for learning mathematics, much of the content described is preserving the status quo of curriculum content typical of Australian states. The notion that mathematical learning begins at five years of age (foundations) does not take into account the rich development of mathematical thinking that occurs prior to, and in the first three years of formal schooling, including the notion that young children are capable of abstract ideas and simple generalisations that can promote algebraic reasoning.

The scope of the number and algebra descriptors is limited and does not reflect current research with 4 to 6 year-olds that shows a significant number of children able to move beyond these basic limits. Thus, the concern is raised that the content described as appropriate for years K-2 does not challenge a significant portion of children in this age range.

Students in the Foundation year sort and classify objects and work with patterns based on observing natural patterns in the world around us, using materials, sounds, movements or drawings. They copy, continue and create patterns with objects and drawings (ACMNA005). In Years 1 and 2 students investigate simple number patterns in the numeration system through activities such as skip counting. They investigate and describe number patterns formed by skip counting and patterns with objects (ACMNA018).

The development of the concept of pattern might have been articulated with depth and continuity. The notion of pattern as a 'unit of repeat' could be made explicit in a sequence extending beyond simple repetition to include complex repetitions and the representation of the structure of patterns, albeit using invented symbolism. Pattern as 'unit of repeat' is interrelated with counting in multiples (or skip counting), repeated addition and early multiplicative thinking; this links with formal multiplication and proportional reasoning. Yet these fundamental concepts lack attention in the content descriptors. The opportunity to include the development of growing patterns, and functional thinking also seems to have been lost despite the fact that exploration of growing patterns supports the development of number patterns. Early algebraic thinking also includes the notion of equivalence but this does not appear until much later in Years 5 and 6. Important relationships such as developing commutativity and associativity are not made explicit which are important in modeling the structure of simple additive number sentences and word problems.

In comparison, the research-based National Council for the Teachers of Mathematics (NCTM) (2006) *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics* provides more explicit descriptions of, and makes direct connections between number and operations and algebra, and other 'focal points' in geometry, measurement and data analysis. Beginning Pre-Kindergarten student expectations are more advanced than for Patterns and Algebra in the *Australian Curriculum: Mathematics*. Number Operations and Algebra are often paired as a Focal Point and Connection. As well, the Algebra content is described separately providing a curriculum map that makes both the connection explicit as well as the specific place of algebra.

In essence, in Pre-K and K, students are expected to analyse how both repeating and growing patterns are generated; in Grade 1, they are required to "illustrate general principles and properties of operations, such as commutativity using specific numbers", and by Grade 2 "develop an understanding of invented and conventional symbolic notations, and describe qualitative and quantitative change" (NCTM, 2006, p. 24).

As early as Kindergarten, the Curriculum Focal Points and Connections describe patterns as "preparation for creating rules that describe relationships" and by Grade 1 "Children use mathematical reasoning…commutativity and associativity and beginning ideas of tens and ones to solve two-digit addition and subtraction" (NCTM, 2006, p. 25). In Grade 2, the interrelationships with number patterns are encouraged: "In Grade 2, through identifying, describing and applying number patterns and properties in developing basic facts, children learn about other properties of numbers and operations" and "develop, discuss and use efficient, accurate and generalisable methods to add and subtract multidigit numbers" (NCTM, 2006, p. 27)

Patterns and Algebra in Years 3–6

By Years 3 and 4 the *Australian Curriculum: Mathematics* content descriptors state that children are to be engaged in describing, continuing and creating number patterns that involve simple numerical operations with whole numbers and they describe patterns with numbers and identify missing elements (AMNA035). They describe, continue, and create number patterns resulting from performing addition or subtraction (ACMNA060) and solve problems by using number sentences for addition or subtraction (ACMNA036) and rational numbers (Years 5 and 6). The use of number sentences to represent word problems and finding unknown quantities is also introduced in Years 4 to 6. By Year 6 students "begin" algebraic thinking by describing the rules used to create number sequences. They continue and create sequences involving whole numbers, fractions and decimals and describe the rule used to create the sequence (ACMNA133). They explore the use of brackets to order operations and to write number sentences (ACMNA134).

Fundamentally, Years 3-6 are critically important for developing and consolidating the early understanding of structural relationships developed in the K-2 years. For example, the following two content descriptors from Number and Algebra in Years 4 and 5 could be used by teachers to extend the fundamental idea of equivalence, which can be introduced very early and is such an important concept for the later years.

> Year 4: Use equivalent number sentences involving addition and subtraction to find unknown quantities (ACMNA083)
>
> Year 5: Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121)

Consider number sentences such as 39 – 15 = 41 – □, or 5 × 18 = 6 × □. Teachers might use examples like these and interpret the content descriptors in one of two contrasting ways. First, the emphasis might be focused on using computation and equivalence to obtain a correct answer. In this situation teachers would encourage students to simplify each number sentence by calculating the value of the known pair of numbers, [24 in the subtraction sentence, and 90 in the case of the multiplication sentence]. They may then ask what unknown number on the right hand side will be needed to give these results, leading to 17 for the subtraction sentence and 15 for the multiplication sentence. It might be possible to some teachers to argue that this is all that is needed. However, this approach to teaching would neglect important opportunities to extend students' understanding of equivalence and its embodiment in different operations.

Second, an altogether richer approach would be to look more deeply at the structure of these and related equivalent number sentences; looking especially at how the direction of compensation changes according to the operations involved. In this kind of approach, students are encouraged to look at the numbers either side of the equivalent sign and to refrain from calculating. Some students may express this reasoning verbally, using rich and varied forms of mathematical thinking such as: "*Because 41 is two more than 39, I have to put a number that is two more than 15 in order to keep the same difference*". Other students may express their thinking by using arrows to connect related numbers, 39 to 41 and 15 to the unknown number, concluding that it has to be two more than 15 "to keep both sides the same". Similar forms of reasoning may be expressed in different forms of words; some students may explicitly use words such as "equivalent" or "to keep both sides the same". In all these cases, students are clear that they are dealing with equivalent differences. These students notice that the direction of compensation used the case of subtraction or difference operates in the opposite way to sentences involving addition. Likewise, for the multiplication sentence, students will be encouraged to notice that since 18 is three times the value of 6, the missing number has to be three times 5 in order to maintain equivalence. Students need to notice that because this sentence is about multiplication, the fact that the 6 is one more than 5 is not important, whereas the multiplicative relationship between 6 and 18 is the key to a solution.

These approaches focus on important and generalisable features of sentences involving the same operations. These features are intended to support students' computational fluency, and also to prepare them for algebraic thinking (Mason et al., 2009). These possibilities are quite new to many Australian primary and junior secondary teachers. We need to ask how well the new *Australian Curriculum: Mathematics* will open up teachers' vision to these ideas.

Another aspect that needs to be addressed is the inclusion of functional thinking. Research shows that young children are capable of thinking functionally. Functional thinking not only illuminates the relationship between the operations and their inverses but underpins much of algebraic thinking in the later years. Even young children have been found to understand situations involving co-variation. This a major gap in the *Australian Curriculum: Mathematics* and overlooking this important aspect in the content for primary students is perhaps the most serious omission in Number and Algebra.

By comparison, the NCTM (2006) *Curriculum Focal Points* Algebra are far more challenging. From Grade 3, "Children use number patterns to extend their knowledge of properties of numbers and operations for examples when they build foundations for understanding multiples and factors" (p. 28). For Grades 3–5 the focal points describes algebra as "understanding properties of multiplication and the relationship between multiplication and division as a part of algebra readiness". Through the creation and analysis of patterns and relationships involving multiplications and division…students build a foundation for later understanding of functional relationships by describing relationships in context with statements such as, "the number of legs is 4 times the number of chairs"(NCTM, 2006, p. 29). The expectations at Grades 3–5 describe algebra as requiring students to describe extend and make generalisations about geometry and numeric patterns and represent and analyse patterns and functions using words tables and graphs; identify properties such as commutativity, associativity and distributivity; represent

the idea of variable as an unknown quantity using a letter or symbol and express mathematical relationships using equations. By Grade 6 they are expected to write, interpret and use mathematical expressions and equations.

The development of mathematical concepts and proficiencies appropriate to Grade 6 requires an understanding of the structure of mathematical concepts, and the relationships between them (Mulligan & Mitchelmore, 2009), and acquisition of processes of abstraction, symbolisation and generalisation that are peculiar to mathematics (Mason, et al., 2009). The proposed content sequence for the *Australian Curriculum: Mathematics* does not do enough to highlight this fundamental idea of structure, in a way that exemplifies meaningful development. The notion that students do not 'begin' algebraic thinking until Year 6 explains to some extent why the content sequence from Foundation to Year 2 is limited. In comparison, the NCTM *Curriculum Focal Points* promotes the development of algebraic thinking from Kindergarten. It makes explicit the importance of the structural development of mathematical relationships, expressions and their connectedness to each other and other concepts as a 'big' unifying idea.

Another issue is that the content descriptors in Patterns and Algebra are developed sequentially, i.e. one after the other when they could be developed simultaneously. For example repetition is developed prior to introduction of equivalence and function. New research evidence demonstrates that several aspects of patterning and early algebraic thinking can be developed simultaneously with important links made between different concepts (Williams, 2010, Recommendation 24).

Algebra in the Secondary Years

It is generally accepted that all students should have access to algebraic reasoning (Yerushalmy & Chazan, 2008) since the ability to reason algebraically is a prerequisite for participation in higher levels of mathematics and it is important for access to many fields of employment that underpin our national economic growth (MacGregor, 2004).

Three decades ago Küchemann (1981) indicated that many students think of an algebraic symbol such as '*x*' as an unknown quantity and few consider the possibility that an unknown symbol can be a variable having multiple values. One reason for considering '*x*' as an unknown number instead of a variable may lie in students' previous arithmetic experience. Bednarz and Janvier (1996) found that difficulties in the transition from arithmetic to algebra stem from the differing nature of problems presented in each field and the various procedures used to solve these problems. In arithmetic calculations, students move from a known quantity to an unknown quantity. However, algebraic problems proceed from unknown to known quantities and are designed to indicate the relationships between the variables.

Students who interpret letters as specific unknowns or unknown quantities rather than as generalised numbers often learn the processes of evaluation and symbol manipulation without assigning any meaning to the letters involved (Booth, 1995). Misconceptions about variables are also linked to concepts like equivalence (Knuth, Stephens, McNeil, & Alibali, 2006; Perso, 1991). Learning and teaching in algebra therefore needs to focus more clearly on developing an understanding of variables instead of giving so much emphasis to symbol manipulation and equation solving (Warren, 2003).

Patterns and Algebra and Linear and Non-Linear Relationships in Years 7-10

In the *Australian Curriculum: Mathematics*, students in Year 7 continue the work begun in the primary years on generalising arithmetic properties and operations (ACMNA177) and the concept of a variable is established as a way of using letters to represent numbers in algebraic expressions and word descriptions (ACMNA175). Students create algebraic expressions and evaluate them by substituting values for each variable (ACMNA176).

The opportunity for students to develop a sound conceptual understanding of variables in Year 7 before they proceed to more abstract symbol manipulation is a positive step forward. However, it seems that students are expected to learn algebraic skills before using them in authentic situations and it may be preferable to adopt a problem-solving approach instead. For instance, students could take a worded problem and tease out the unknown quantities and the mathematical operations that link them to arrive at a solution without the need for formal algebra. Perhaps this could be more usefully done using a spreadsheet where the process of entering operations into the cells becomes a precursor to algebraic thinking. In this way, students will be better able to appreciate where the algebra they are leaning is applied.

In Year 8, students learn to expand (ACMNA190), simplify (ACMNA192) and factorise (ACMNA191) algebraic expressions based on simple operations and numerical factors. The notion that the laws applying to numbers can be generalised using variables is also reinforced. The laws for variables involving positive integral indices and the zero index are introduced in Year 9 (ACMNA212) and there are some good links made from the algebraic indices to number work in Indices and Scientific Notation in the real numbers sub-strand and to simple interest calculations in the money and financial mathematics sub-strand.

Students also work with binomial products and learn to collect like terms in algebraic expressions (ACMNA213). Factorisation is extended in Year 10 through

work with algebraic factors (ACMNA230) and index laws are used to simplify algebraic products and quotients (ACMNA231), leading to the need for negative indices. Students in Year 10 also study simple algebraic fractions (ACMNA232), extend their work with binomial products to investigate special binomial products based on perfect squares and the difference of squares, and they learn the method of completing the square to solve monic quadratic equations (ACMNA233). The content associated with substitution into algebraic expressions is also extended to substitution into formulas to determine the value of an unknown quantity (ACMNA234). The algebra for Year 10A covers polynomials and includes algebraic long division and the factor and remainder theorems (ACMNA266).

The Linear and Non-Linear Relationships sub-strand begins in Year 7 with an introduction to the Cartesian plane which focuses on plotting points with given coordinates and on finding the coordinates of a given point (ACMNA178). Simple linear equations are solved using concrete materials, the balance method, and strategies such as backtracking, guess and check and improve (ACMNA179). Students also investigate travel graphs and interpret other straight-line graphs based on real-life data.

In Year 8, students plot linear relationships with and without the use of technology (ACMNA193), learn to solve linear equations using algebraic and graphical techniques, and continue their work solving linear equations using a variety of methods (ACMNA194). Distance (ACMNA214), midpoint and gradient (ACMNA294) are introduced in Year 9 using a range of approaches, including graphing software. Students sketch linear (ACMNA215) and simple non-linear relations such as parabolas, hyperbolas and circles (ACMNA296), with and without the use of technology. They also determine the equations of lines from tables of values and graphs.

In Year 10, students solve problems based on linear equations (ACMNA235) and learn about linear equations involving simple algebraic fractions (ACMNA240). They solve quadratic equations and identify the link between the x-intercepts of the parabola and the roots of the corresponding quadratic equation (ACMNA241). Equations are used to represent word problems and students learn how to solve linear inequalities, graphing the solutions on a number line (ACMNA236). Linear simultaneous equations are introduced using algebraic and graphical means to solve them (ACMNA237). Co-ordinate geometry is extended through an investigation of the properties of parallel and perpendicular lines (ACMNA238) and an exploration of the connection between algebraic and graphical representations of simple quadratics, circles and exponentials (ACMNA239).

The algebra content for Year 10A focuses on work with non-linear graphs such as parabolas, hyperbolas, circles and exponentials to study transformations of these graphs by connecting their graphical and algebraic representations (ACMNA267). The work on equations is extended to simple exponential equations based on population growth (ACMNA270), as is the polynomials topic through sketching a range of curves (ACMNA268), investigating the features of these curves using technology, and solving monic and non-monic quadratic equations by factorising (ACMNA269). However, it is questionable whether the extra algebra content covered in 10A will be sufficient preparation for those students who intend to study mathematics at the highest level in Years 11 and 12 since a solid foundation in algebra is essential when learning advanced mathematical topics.

Research on Technology use in Patterns and Algebra and Linear and Non- Linear Relationships

More than a decade ago, Clements (1999) highlighted the benefits of virtual manipulatives for classroom use. For example, virtual Pattern Blocks have colours that can be changed and they can be 'snapped' into position unlike concrete material and they "stay where they're put" (Clements, 1999, p. 51). The role and benefits of software and virtual tools, such as *Kidpix* and *Kidspiration* (Hong & Trepanier-Street, 2004) and virtual manipulatives (Highfield & Mulligan, 2007; Moyer, Niezgoda & Stanley, 2005) show that each of these tools has potential advantages for developing patterning skills. Yet there is scant attention to the integration of these readily available resources to support the development of content and acquisition of curriculum proficiencies such as understanding and reasoning.

When using virtual manipulatives and dynamic interactive software it appears that making judicious and wise choices in software selection is key to mathematical success. For example while there is a large range of virtual manipulatives, only some, such as the 'Pattern Block' manipulatives are open-ended and have considerable mathematical potential. Other virtual manipulatives, such as 'Complete the Pattern' are not as powerful mathematically as they offer a closed questioning style and do not allow transformation or manipulation that is necessary for early mathematical development.

In the secondary school, technology can significantly change the nature of the opportunities for algebraic activities of conceptualising, generalising, and modeling (Thomas, Monaghan, & Pierce, 2004). There has been considerable research on the role of technology in enhancing students' development of the function concept and their ability to confidently apply processes of symbol manipulation (Kieran & Yerushalmy, 2004).

Spreadsheets provide an intermediate step between arithmetic and algebra, especially in assisting students to develop their understanding of a variable as

something that varies (Haspekian, 2005; Yerushalmy & Chazan, 2008), and they allow students to begin their study of functions using tabular representations (Confrey & Maloney, 2008). The dynamic and interactive nature of graphing software can support students' control of the appearance of a graph (Ainley, 2000) so that they can explore many features of graphs and are more likely to investigate hypotheses to discover new relationships (Ruthven, Deaney, & Hennessy, 2009). Graphing software also affords both local and global views of graphs, and it allows students to explore multiple representations to highlight different graphical features (Heid & Blume, 2008). Computer Algebra Systems (CAS) permit a richer treatment of functions, both as processes and as mathematical objects in their own right. Students can use CAS to make connections between graphic, numeric and symbolic representations of functions, and reason about them (Zbiek & Heid, 2008).

Access to CAS tools that allow automatic generation and operation on symbolic expressions highlights the need for a reconceptualisation of the role of symbol manipulation (Yerushalmy, 2000; Noss, 2001). For instance, there could be greater emphasis placed on word problems and practical applications rather than solving symbolic equations (Nunes-Harwitt, 2004/05). CAS and other symbol manipulation programs will necessitate new emphases, such as the need to deal with equivalence of symbolic expressions (Artigue, 2002). At the same time, new difficulties for students may arise, such as the need to distinguish variables from parameters (Drijvers, 2003).

Technology in Patterns and Algebra and Linear and Non-Linear Relationships Sub-strands *The Australian Curriculum: Mathematics* advocates that mathematics classrooms will make use of all available ICT in teaching and learning situations with the expectation that a developmentally appropriate range of technological resources might be aligned with the content descriptors and achievement standards (Goos, this volume).

Although there is some reference to the use of technologies for learning in the Number strand it is omitted from the Patterns and Algebra sub-strand from Foundation to Year 7 and is presented as optional rather than a mandatory requirement in Years 7 to 10. The integration of technologies could have enhanced the content by providing links to a rich range of technological tools and resources explicitly supporting the implementation of Patterns and Algebra. Exemplars of web-based resources, digital learning objects and other software from the pre-Foundation years might have highlighted the affordances and limitations of these technologies for mathematics learning.

In the Patterns and Algebra sub-strand, patterning forms a 'thread' where children engage in simple through to complex patterning, primarily developed as repetition. Generally these patterning experiences imply the use of concrete materials and representations of patterns drawn or made by the children using traditional media. Without explicit reference to the use of digital resources in developing patterning and early algebra concepts this curriculum does little to encourage teachers and students to maximise opportunities to create or represent patterns in an on-screen environment.

Using Technology in Years 7-10

Guiding students' development of algebraic reasoning in a technology-rich environment will therefore require a curriculum that gives more prominence to the structure and key features of algebraic expressions. Students will need to correctly enter expressions into a CAS, efficiently scan the working and results to identify any possible errors, and interpret the output. Artigue (2002) has developed the notion of *technique* to describe a more expanded view of symbolic manipulation required in a CAS environment. Similarly, Pierce and Stacey (2001) highlight the importance of *algebraic insight* or the mathematical thinking students need to do algebra with CAS, including algebraic expectation and the ability to link representations. Similarly, whilst technological tools support generalisations, they also affect the nature of those generalisation activities (Hershkowitz, Dreyfus, Ben-Zvi, Friedlander, Hadas, Resnick, Schwarz, & Tabach, 2002). For instance, the rapid generation of a large number of examples using technology can afford students greater opportunities to identify patterns and generalise their results, such as when observing how the changes in values of parameters of functions can affect their graphs.

The availability of technological tools can serve as a powerful catalyst for rethinking the traditional algebra curriculum in the secondary years (Stacey, Asp, & McCrae, 2000). For example, CAS facilitates a more generalised approach to the teaching of equations (Ball, 2001) and it can support the use of different types and combinations of functions that permit modelling of real life situations (Thomas, 2001). In these activities, symbolic reasoning assumes greater importance than manipulation and simplification of algebraic expressions (Heid, 2002; Heid & Edwards, 2001) since CAS coupled with dynamic construction tools allows students to link symbolic results to numerical and graphical representations rather than focus so heavily on abstract symbol manipulation.

Technology may provide more access to algebraic thinking and techniques for a greater number and variety of students, but can also present barriers for other students (Thomas, Monaghan, & Pierce, 2004). Hence, there is a need for algebra curriculum changes in the order of topics, their relative importance, and the way they are presented (Yerushalmy, 2005). In particular, more numerical solution methods for certain types of problems and realistic problem solving (as opposed to learning symbol manipulation techniques before trying to apply them) should be

emphasised. The introduction of technology facilitates investigation of multiple representations and also has the potential to change the way some aspects of algebra and functions are introduced. But care needs to be exercised since, for example, a sequential treatment of symbolic and graphical representations does not assist students in establishing links between them. Finally, there remains the question of how much algebra should continue to be done by hand and how much can be realistically achieved using technology (Flynn, Berenson, & Stacey, 2002).

The role of digital technologies and graphing software in the learning of algebra in Years 7 to 10A is somewhat problematic. Digital technology should be used when it improves teaching and learning efficiency. A static display, even one that results from student inputs, rarely adds meaning for the student whereas a dynamic display that is manipulated by student input is more likely to be central to the learning of new concepts. There are now emulators of graphic display calculators that can be installed on a computer to offer the added benefit of immediate feedback. Students and teachers can connect their calculator display to a digital projector so all members of the class can view it for the purposes of discussion and analysis.

Technology is not mentioned at all in the patterns and algebra sub-strand when it might have been advantageous to include technologies in order to strengthen students' ability to link numerical and symbolic representations of algebraic expressions. Technology is highlighted in many of the outcomes for Years 8 to 10A of the linear and non-linear relationships sub-strand through activities such as plotting graphs, solving equations by graphical methods, and connecting graphical and algebraic representations of functions. However, the approach adopted in recommending the use of technology is rather cautious and the use of technology could have been made more explicit. For instance, phrases such as "using a range of strategies, including graphing software" or "with and without the use of digital technologies" may suggest to teachers that the use of technology is supplementary to their needs rather than a central focus of classroom learning and teaching.

The explicit use of technology is not always articulated in the content descriptors. For example, Williams (2010) raises the question of why technology is not exemplified in the exploration of the relationship between quadratic functions and their graphical representations in Year 10. Given that students now have ready access to spreadsheets, graphic display calculators, and other computer software, it is questionable whether to wait until Year 10 before linking problem solving and algebra in a meaningful way. It appears that a very traditional approach to learning algebra underlies the *Australian Curriculum: Mathematics* and that digital technology is seen as something of an accessory.

Interrelationships Between Number and Algebra and the Proficiencies Strand

The proficiency strands of Understanding, Fluency, Problem Solving and Reasoning describe the actions in which students can engage when learning about the syllabus content. In the early sections of the chapter we have discussed various aspects of mathematical understanding connected to structural relationships between arithmetic and algebraic reasoning. The Problem Solving and Reasoning proficiencies are also of particular relevance to algebra, as is Fluency is terms of mental calculation and efficient computational strategies.

A focus on equivalence and patterning in the primary school years should have an immediate pay-off for fluency in calculation, including mental computation. For example, adding or subtracting 9, 99, 999 (as well as other numbers close to these, and using 'near decade' numbers such as 39, 49) is made easier and more fluent by knowing how sentences involving these numbers can be converted, for example, into equivalent or near equivalent sentences involving 100 and so on. A similar case can be made for multiplication and division.

Problem solving is the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively (ACARA, 2011). Students apply their mathematical understanding by planning how to solve unfamiliar problems and checking the reasonableness of their answers. In the primary years, problem solving is described as the formulation of problems based on authentic situations, the use of materials to model problems, and using number sentences to represent the problem situations. In the secondary years, problem solving is aligned more closely to specific content such as using algebraic and graphical techniques to solve simultaneous equations and inequalities in Year 10.

The direct link between Number and Algebra and the Reasoning proficiency strand should enable students to develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising (ACARA, 2011). Reasoning activities that relate to the algebra content thus involve creating and explaining patterns, and generalising number properties. Students are expected to transfer their mathematical understanding among different situations and develop notions of mathematical proof.

However, the development of informal proof is given superficial attention and could be developed as a 'bottom-up' strategy from the early years. It would be difficult for a teacher to trace the introduction and development of 'proof' in the Patterns and Algebra sub-strand year by year. The processes of justification and argumentation are integral to the notion of mathematical proof from the preschool years (Perry & Dockett, 2008), yet they are not described or linked to the content

descriptors. The language of reasoning and proof may present particular challenges for students in the primary years. That is why the distinction between when it is appropriate to use the term 'prove' and terms like 'justify', 'explain', 'support' 'demonstrate' or 'convince' needs to be provided. If the curriculum is serious about developing students' skills in justification, argumentation and proof in preparation for more formal mathematics then these proficiencies need to be linked explicitly to Patterns and Algebra and assessed both informally and in assessments such as the National Assessment Plan Literacy and Numeracy (NAPLAN).

Challenges Facing Teachers and the Impact of Patterns and Algebra on the Development of Teachers' Pedagogical Content Knowledge (TPCK)

Some may even view the introduction of the pairing of algebra with number strands as difficult to align and integrate. The pairing of these content strands seems reasonable given that some categorisation of content will be necessary. But for primary teachers who have traditionally taught the number strand and its sub-strands focused on counting, place value computations, and fractions and proportional reasoning, the pairing with algebra may be difficult to conceptualise. What might be even more difficult is to trace the development of arithmetic structural relationships in terms of early algebraic reasoning from the early grades.

An experienced upper primary teacher questioned recently whether the Number strand had been given less emphasis to make way for algebra on the assumption that the Number content had been 'halved' to incorporate algebra. Moreover for teachers in the early grades, it is challenging to conceptualise what is meant by teaching early algebra. Traditionally this has been restricted to the development of patterning skills.

In particular the new curriculum presents challenges for teachers in the early and primary grades. The notion of unit of repeat, growing patterns, functions and equivalence are but some of the big conceptual ideas that required robust understanding in order to implement effective pedagogical strategies. In turn this will have implications for the transition from arithmetic to algebra in the secondary school.

Implications for Curriculum Implementation and Further

Within the last decade or so, a more coherent research base has evolved to inform the re-development of a scope and sequence from preschool through to Year 12 to incorporate early algebraic reasoning, and algebra within mathematics curriculum (Blanton & Kaput, 2003; Clements & Sarama, 2009; Jacobs et al., 2007; Mulligan et al., 2011; Papic et al., 2011; Warren & Cooper, 2008).

A number of new classroom-based studies are looking at ways that teachers can promote the development of early algebraic reasoning, structural relationships and generalisation in children's early mathematics learning. Research on effective professional development for pre-service and practicing teachers must be integral to this work. Two questions that the research raises are how teachers can develop a deeper understanding of why pattern and structure is critical to early algebraic thinking, and how they may overcome traditional perceptions that algebra is the exclusive domain of secondary school mathematics. "Some may react to the idea of introducing algebra in the elementary school with puzzlement and skepticism" (Carraher & Schliemann, 2007, p. 670). Another confounding problem that needs urgent research may be lack of teacher pedagogical content knowledge particularly for the early and primary professionals (Ball et al., 2008). More research is needed to identify underlying weaknesses in teacher pedagogical content knowledge and the development of relational thinking necessary for the teaching of algebra (Asquith et al., 2007).

In our discussion we have provided an outline of the *Australian Curriculum: Mathematics* Number and Algebra content strand, which does to some extent present a connected view of mathematics, by linking number with pattern for algebraic thinking. On the surface this provides a positive sense of expectation about the Number and Algebra strand, with the opportunity for greater conceptual and connected knowledge and the development of teaching practices that focus on relational thinking. The definitions of the Proficiency strands (understanding, fluency, problem solving and reasoning) indicate a curriculum that is concerned about mathematics as patterns, relationships and generalisations, and not just facts, skills and rules. However, the structure and depth of the content, as presented through the descriptors appear to lack the coherence and connectedness that the document promises, which results in some concerns about the implementation and effectiveness of the strand overall.

In grappling with the implementation of the *Australian Curriculum: Mathematics* further fundamental questions are raised: What is the appropriate content and sequence for teaching algebra in the K-6 (primary) school that students are capable of achieving, and what is its relative importance in the Number and Algebra strand across all grades?. What relation does it have to the other two strands–Measurement and Geometry and Statistics and Probability? Moreover is early algebra really about developing proficiencies in reasoning, abstracting and generalising? What role does the early introduction of algebra play in the mathematics curriculum overall? What forms of assessment of early algebra, beyond patterning skills, will students in the primary years encounter i.e. NAPLAN, standardised assessments? How will the new Number and Algebra scope and sequence influence the way teachers approach

traditional pedagogy focused on number skills and operations? How will teachers come to change their practice to incorporate the wide range of technological tools available for the contemporary teaching of number and algebra in the 21st century?

What we may need in the *Australian Curriculum: Mathematics* is not merely adding early algebra to the number content strand for the primary school years, or beginning algebra earlier in the mid or upper primary grades. Early algebraic reasoning is not formal algebra introduced earlier. What is perhaps needed is a reconceptualisation of the development of algebraic thinking and the interrelationships between algebra and arithmetic and other aspects of mathematics. Only then can curriculum developers weave it appropriately throughout the mathematics curriculum.

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