Testing for Additivity in Intuitive Thinking of Area

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Many students confuse area and perimeter and are additive in their thinking of area. This misconception affects their ability to successfully complete area estimation and calculation tasks. In this paper, a simple pen-and-paper test based on Kidman's (1999) research into area judgement rules is developed and trialed and used to detect additive (and multiplicative) thinking of area estimation. It found that approximately one half of the students' intuitively judged area additively showing the power and persistence of this form of thinking.

The relation between area and length is an aspect of area that causes students difficulty, particularly with comparing and estimating areas. Area relates to length in terms of multiplication. This is most evident in rectangular shapes where doubling the length of each side will result in a rectangle with an area that is 4 times larger. Many students find this difficult to comprehend (Kidman & Cooper, 1996) because they perceive area in terms of the boundary of the region, that is, the shape. They tend to confuse perimeter with area and have an additive view of the relationship between length and area (Anderson & Cuneo, 1978).

The consequences of believing area is additive can be seen in the graphs of perimeter and area for rectangles where sides are doubled (Figure 1). Consider, the four rectangles below:

Rectangle A	2 cm by 3 cm	Area = 6 cm^2; perimeter = 10 cm
Rectangle B	2 cm by 6 cm	Area = 12 cm^2; perimeter = 16 cm
Rectangle C	4 cm by 3 cm	Area = 12 cm^2; perimeter = 14 cm
Rectangle D	4 cm by 6 cm	Area = 24 cm^2; perimeter = 20 cm

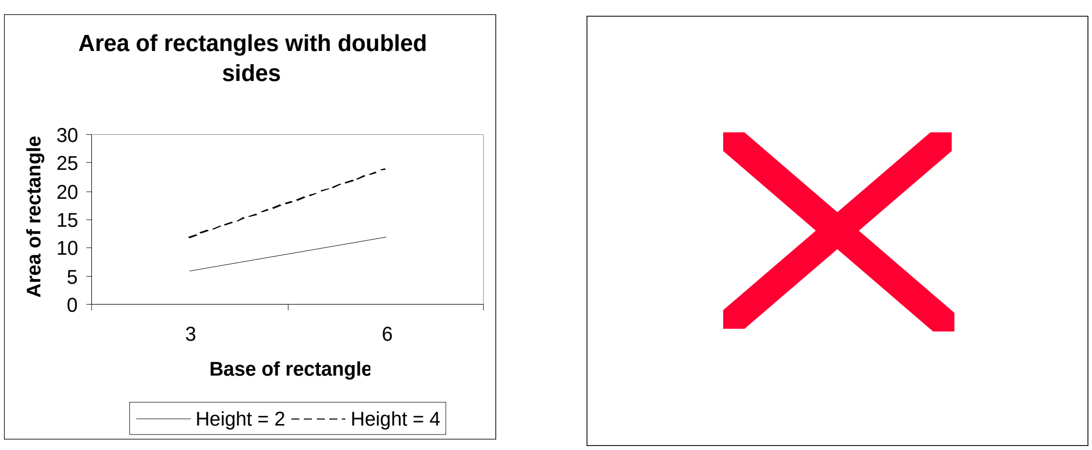


Figure 1. Areas and perimeters of rectangles with doubled sides.

As is evident in Figure 1, the perimeters of these rectangles form parallel lines while the areas form diverging lines. This difference between perimeter and area has been used to develop a research method for testing whether students' intuitive perceptions or estimations of area of rectangles are additive or multiplicative (e.g., Anderson and Cuneo, 1978; Kidman, 1999). Studies using this method have shown that many students maintain an additive perception of area into their secondary and tertiary years. The particular persistence of additive perceptions of area was found to affect not only area estimation and comparison but also the type of techniques and strategies used for considering rectangular area and completing the *Area Calculation Tasks* in Figure 2 (Kidman, 1997, 1999, 2001). Additive thinkers generally did not attempt the *Area Calculation Tasks.* The few that did concentrated on the boundaries of the regions. On the other hand, multiplicative thinkers were generally able to solve all the *Area Calculation Tasks*, using diagrams, drawing subdivisions and skip counting as they went.

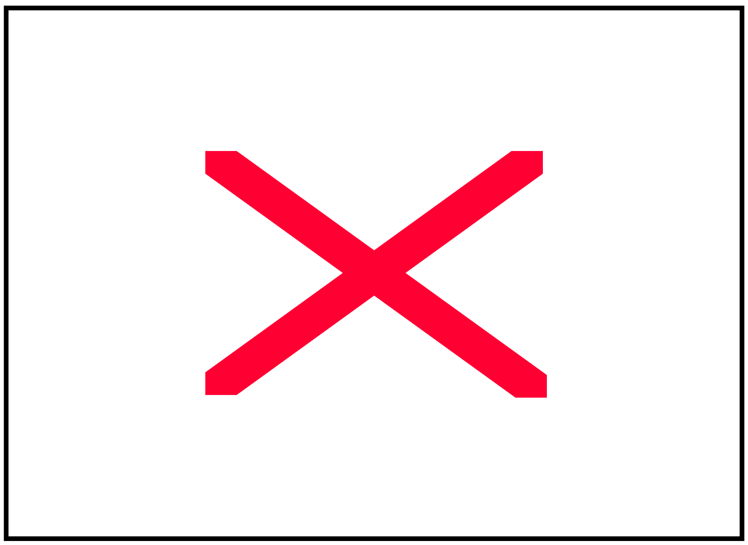


Figure 2. The Area Calculation Tasks.

Kidman (2001) found that the research method categorised students as predominantly additive, predominantly multiplicative, or transitional. Further, she found that predominantly additive and multiplicative students employed different strategies (as in Figure 3) when presented with area estimation and the *Area Calculation Tasks*, while transitional students employed a mixture of additive and multiplicative strategies, focusing on perimeter but also using 1-1 counting. Transitional students seemed to have the ability to think of units of one, and of units of more than one, but not both simultaneously. Like additive students, they tended to confuse area with perimeter.

The high number of students who are transitional or additive with respect to area is a major problem in teaching mathematics. Without detection and remedial instruction, these students will not be able to understand or apply area knowledge, one of the basics for employment and life, in real world situations. This paper focuses on the development and trial of: (a) a pen-and-paper test, easily implemented by classroom teachers, that identifies additive and transitional students with respect to area; and (b) classroom activities, again easily implemented by classroom teachers, to remediate area understandings of transitional and additive students.

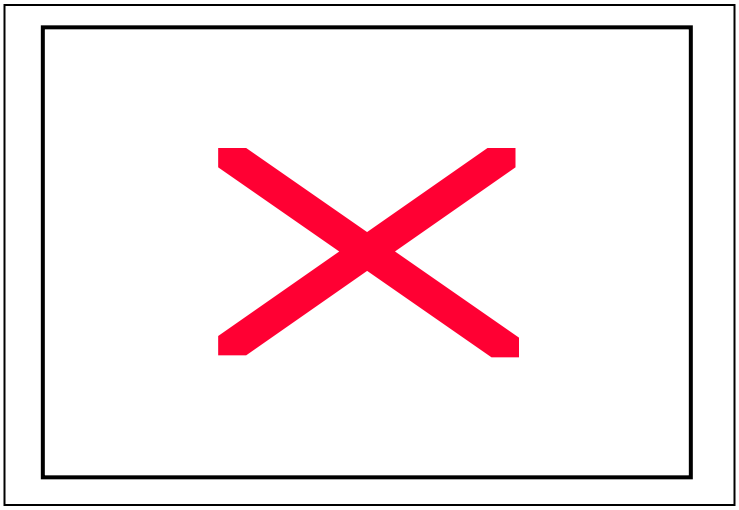


Figure 3. The links between intuitive thinking and estimation and calculation tasks (Source: Modified from Kidman, 2001)

Development

The research method for identifying additive and multiplicative thinking used in Kidman (1996, 1997, 1999, 2001) and was very time consuming as it used individual interviews of students and many tasks. To overcome this time factor, the method was modified to create a simple pen-and-paper test.

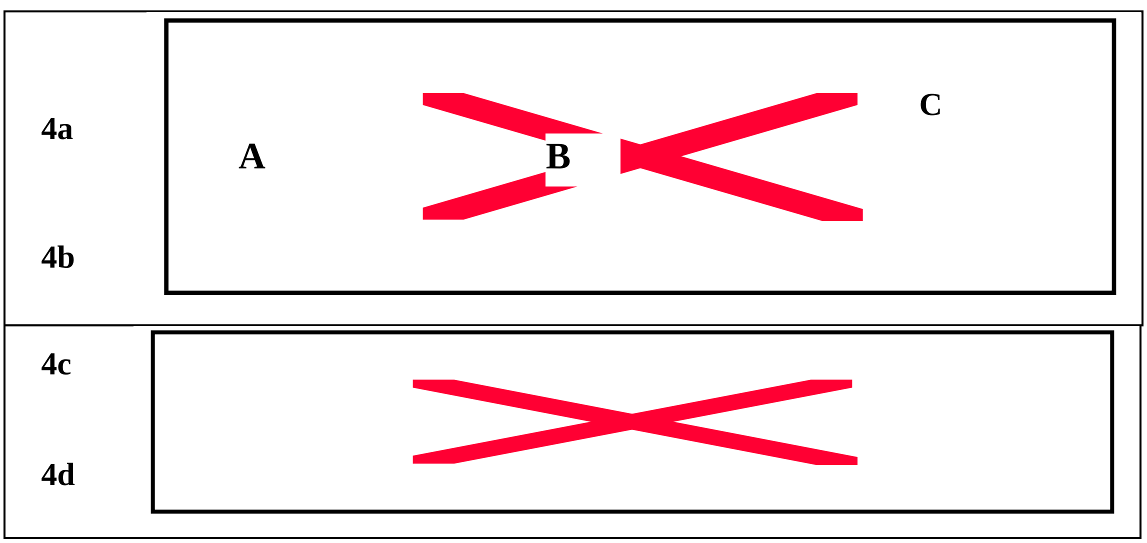
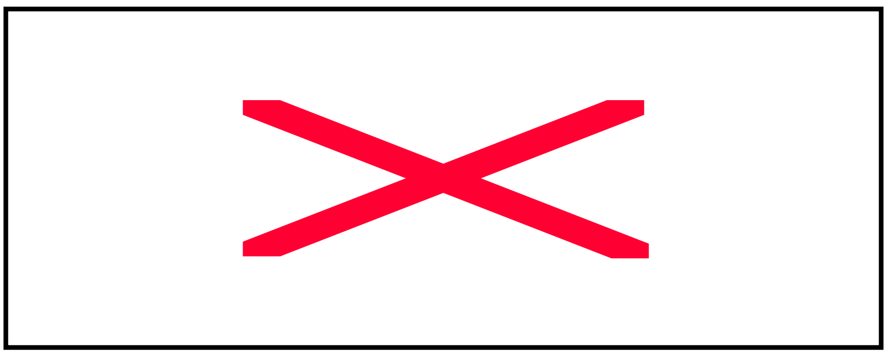


Figure 4. (a) Given rectangular regions, (b) given line segment, (c) additive response, and (d) multiplicative response

This difference became the basis of a pen-and-paper *Area Additivity Test* for area judgement thinking developed by Kidman & Cooper (2001). The students were presented with diagram 4a and line segment 4b as in Figure 4. The students were additive id they marked the areas of B and C as per 4c while they were multiplicative if they marked as per 4d. That is, the additive students evenly spaced there positions for A, B and C, while the multiplicative students had a larger gap between C and B than between B and A.

This difference between additive and multiplicative thinking with respect to area was also the basis for the classroom activities to remediate additivity. The first activity was aimed at investigating the relationship of area to perimeter. The students were each given 12 flat, square tiles, and asked to make as many different shapes as they could from all 12 tiles. This activity held area constant, while varying perimeter. The second activity again used the same tiles, and had the students constructing a rectangle from 3 tiles (see Figure 5a). The students investigated the area and perimeter of the rectangle that had one of its dimensions doubled (see Figure 5a) and then both dimensions doubled (see Figure 5b).

5a



5b

Figure 5. One dimension doubled (5a) and both dimensions doubled (5b).

Method

The *Additivity Test* was trialed by comparing its findings with those from the *Area Calculation Tasks*. As the *Area Calculation Tasks* were used in the Kidman studies (e.g., Kidman 1999, 2001), the responses for these tasks, in terms of correctness and strategies, have been classified in terms of additive, multiplicative, and transitional thinking. The classroom activities were evaluated by determining their effect they had on students' responses to the *Additivity Test* and the *Area Calculation Tasks*.

Subjects and instruments. The subjects of both trials were a Year 6 class of 25 students. The instruments were the *Additivity Test* and the *Area Calculation Tasks*. The *Additivity Test* consisted of eight pen-and-paper items similar to 4a and 4b in Figure 4. The eight sets of rectangular regions used are provided in Table 1. The students were asked to mark where the areas of the unmarked rectangles would be placed on the line segments.

Procedure. The procedure followed was to first administer the *Additivity Test* followed by the *Area Calculation Tasks* to the students in a whole class context under test conditions. The two instruments took less than an hour for the weaker students to complete. The students were requested to show how they attempted each of the *Area Calculation Tasks*. In the week following the administration of the instruments, the students were taught the two classroom activities by their classroom teacher. The focus of instruction was on exploring how changing a region's attributes affects certain measurements. At the end of this instruction, the *Additivity Test* and the *Area Calculation Tasks* were readministered.

Table 1
Rectangular Regions in the Additivity Test

Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7	Set 8
A 1 × 2	D 3 × 3	I 4 × 3	L 4 × 2	O 5 × 4	R 2 × 3	U 3 × 5	X 6 × 3
B 1 × 4	E 3 × 6	J 4 × 6	M 4 × 4	P 5 × 8	S 2 × 6	V 3 × 10	Y 6 × 6
C 2 × 4	F 6 × 6	K 8 × 6	N 8 × 4	Q 10 × 8	T 4 × 6	W 6 × 10	Z 12 × 6

Note. All measurements are in centimetres.

Analysis. For the *Additivity Test*, students were classified as additive thinkers if their marks on the line segment were regularly spaced, (see Figure 4c), multiplicative if the distance between the last two marks was approximately double the distance between the first two marks (see Figure 4d), and transitional if somewhere in between. Over the 8 items of the test, the students were classified as predominantly additive if they were additive for all but one item, predominantly multiplicative if they were multiplicative for all but one item, and transitional otherwise. For the *Area Calculation Tasks*, students were classified as additive, multiplicative or transitional depending on the correctness of their responses and the strategies evident in their responses using the results from Kidman.

Results and Discussion

All students' responses were easily classified as additive or multiplicative in each of the items of the *Additivity Test*. This was somewhat of a surprise, particularly for multiplicative thinking; however students seemed to easily see the multiplicative relationship. As one said: "it is obvious that the big rectangle (T, 4 x 6 cm) is heaps longer and fatter than the others (R and S, 2 x 3 and 2 x 6 respectively), so its marker needs to be heaps away on the line segment."

The classification of students within and across the eight items is shown in Table 2. The predominant classifications were the same at both sittings: 13 multiplicative, 2 transitional, and 10 additive.

Table 2
Number of Students at Each Level of Additivity in Both Sittings of the Additivity Test

	Number of additive classifications								
	0	1	2	3	4	5	6	7	8
Sitting 1	10	3	1	0	0	1	0	1	9
Sitting 2	11	2	2	0	0	0	0	1	9

The students were also classified as additive, multiplicative or transitional according to their responses (correctness and strategies) to the *Area Calculation Tasks*. As can be seen in Table 3, the classifications for the *Additivity Test* and the *Area Calculation Tasks* were the same in nearly all cases.

The interest directed at the *Additivity Test* was: Is this pen-and-paper test capable of detecting additive, multiplicative and transitional modes of area thinking? The responses in both sittings of the test and the congruence of results with those of the *Area Calculation Tasks* revealed that the test was capable of accurately delineating additive from multiplicative thinking. As is shown in Table 2, this clear delineation was across the 8 items; ten students were clearly additive, 13 students were clearly multiplicative and only 4 students were transitional (showing a mixture of additive and multiplicative line segment responses). As well, the consistency of results showed that the *Additivity Test* could have as low as 3 items and still be valid.

Table 3
Links Between Area Calculation Tasks and Predominant Area Thinking for Additivity Test

Sitting 1	Sitting 2
Additive	
8 additive - Concentration on boundary for 1 to 1 counting	8 additive - Concentration on boundary for 1 to 1 counting
7 additive - Perimeter or half perimeter calculation; 1 to 1 counting	7 additive - Perimeter or half perimeter calculation; 1 to 1 counting
5 additive - Perimeter or half perimeter calculation; 1 to 1 counting	
Multiplicative	
2 additive - Skip counting - groups of 5	2 additive - Skip counting - groups of 5
1 additive - Skip counting - groups of 5; draw a diagram; area calculation	1 additive - Skip counting - groups of 5; draw a diagram; area calculation
0 additive - Skip counting - groups of 5; draw a diagram; create subdivisions; area calculation	0 additive - Skip counting - groups of 5; draw a diagram; create subdivisions; area calculation

An interest directed at the classroom activities was: Are these activities capable of changing additive students' area thinking? The results in Table 2 include a response change between sittings 2 and 1 for only two students, and this change did not result in a change to predominant classification. One student (Charlie) had five additive classifications in sitting 1 but only two additive classifications in sitting 2, while another student (Michael) had one additive classification in sitting 1 and no additive classifications in sitting 2 (in discussion with Michael, it became evident that his one additive response was a careless error). The results in Table 3 also show that strategies for the *Area Calculation Tasks* did not change between the sittings of the test (these strategies closely resemble those found by Kidman, 2001 - see Figure 2). Therefore, the two classroom activities were not effective as taught in this study in changing additive thinking.

However, there was one indication, Charlie, that the activities had potential. Charlie did change from additive to multiplicative in some of the *Additivity Test* items and *Area Calculation Tasks*. His responses were very much directed by multiplicative strategies in the second sitting. He initially drew individual units on the perimeter, but completed the grid with a series of vertical and horizontal lines, forming the rows and columns. As he explained: "I used to think area was about the size of the edges … now I think it is about the number of squares in the whole shape … all need to be counted, not just the edge ones". Charlie claimed to have come to this conclusion because of the "table we did a while ago where we counted squares and edge bits". This was in the first of the two classroom activities where the students held the areas constant but varied the perimeters, and documented the areas and perimeters in a table.

Thus, the failure of the classroom activities to change additive thinking may be due to the amount of time spent on the activities not the activities themselves. It appears that the single activity involving 12 square tiles may not be enough to assist students in their multiplicative understanding of area. With respect to this study, this benefit was only evident in one transitional student. Additivity appears to be a particularly robust misconception that may need repeated activities to affect it.

Conclusions and Implications

The findings from this study and the previous studies by Kidman indicate that it is the multiplicative nature of area that forms the basis of understanding and applying area. The studies have shown that a significant percentage of primary and secondary students (almost 50%) do not perceive area as multiplicative, but as additive. This study has shown that it is possible to detect additive thinking with a simple pen-and-paper test, but it has not shown that additivity can be alleviated by a simple teaching program focusing on the multiplicative relationship between area and length. This is unfortunate because additive students will have difficulty with area, a basic mathematics skill that is important in everyday life and work as well as mathematics. Therefore, to alleviate the robust additive misconception, explicit teaching of the multiplicative aspect of area needs to be included throughout all stages in the development of area. Further research is warranted to determine the exact nature of classroom activities that are capable of altering students' additive thinking.

The National Council of Teachers of Mathematics' (2000) Curriculum Guidelines has called for a coherent curriculum where mathematical ideas are linked and built on one another. It argued that this would facilitate understanding, deepen knowledge and expand application. It is evident that this coherence is not present in present curricula, at least, in as they are put into practice. It requires teachers to organize concepts to form fundamental ideas as an integrated whole. This is not an easy task, "and there are no easy recipes for helping all students learn or for helping all teachers become effective" (National Council of Teachers of Mathematics, 2000, p. 17).

Most mathematics-education texts (e.g., Booker, Bond, Briggs & Davey, 1997) and mathematics curricula (e.g., Department of Education, 1987) propose a 5-stage sequence for introducing measures such as area. The first stage is the identification of the attribute of area (i.e., the meaning of area), while the second stage focuses on comparing areas directly and indirectly without using units. The third stage is crucial in that it uses non-standard units to introduce the notion of unit as a way of applying number, a concept developed from discrete objects, to the continuous attribute area. The fourth stage is the introduction of the formal metric units. The difficulties here are the differences between language and symbols (e.g., "square metres" and m^2) that lead to confusion between "square metres" and " metres squared" (Baturo & Nason, 1996), and the size of conversions (e.g., 10 000 cm^2 in 1 m^2). The fifth and final stage covers the common formulae for area (i.e., rectangle, triangle and circle). These formulae relate area to length.

Many of the difficulties students have with area that have emerged from testing programs have appeared to reflect a lack of understanding of the early stages of the area sequence. The understandings behind each of the 5 stages has, therefore, come to be considered part of what it is to fully understand area (e.g., McLean, 1989; Baturo & Nason, 1996); that is, knowing area means being able to identify the attribute, compare without units, use non-standards, know the standard units, and use formulae. Therefore, ways must be found for the multiplicative nature of area to be integrated into each of these five stages.

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