Quadratic Equation Representations and Graphic Calculators: Procedural and Conceptual Interactions

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In spite of considerable research algebra remains a difficult subject for many students. Classroom teaching approaches still often rely on presentation of one or two procedures based in a single representation for solving a given problem type, often the symbolic algebra domain. The aim of this study was to investigate the value of using a multiple-representational environment in which students interact conceptually with graphic calculators (GCs) while solving quadratic equations. While students gained overall from using the GCs, it has been difficult to show specific representational benefits. However, the study has enabled us to describe and exemplify more clearly the nature of some representational interactions.

Acquisition of knowledge is a process that involves constant interaction between the learner and his environment. In mathematics this environment comprises different ways of presenting corresponding ideas or concepts. These may make use of metaphor (e.g., Nunez, 2000) or in other ways represent concepts which are mathematical processes, objects or statements, etc. While the word *representation* has had various interpretations, sometimes due to differing contexts or theoretical perspectives, Kaput (1987, p. 23) describes a representation as involving “two related but functionally separate entities…the representing world and the represented world” and describes four broad types of representation that this idea accommodates, namely cognitive and perceptual; explanatory, involving models; within mathematics; and external symbolic. In later papers, Kaput (1989, 1998) refers to a representation system as a correspondence between two notation systems (for example equations, graphs and tables of ordered pairs) and uses the terms representation system and notation system interchangeably. In the present study we align ourselves with this view of representation proposed by Kaput.

It is reasonably clear that the cognitive structure of the individual will strongly influence the person’s interaction with an external representation. Kaput (1989) draws a distinction between the way in which students interact with notation systems, describing how some are used mainly to display information and relationships *(display notations)* while others support a variety of transformations and other actions on their objects *(action notations).* Thomas & Hong (2001) also describe how this interaction can take several different forms, including surface and deep observation of representations, and actions performed on them, suggesting that the purpose of a representation is student initiated rather than inherent. They define the idea of a *conceptual representation tool* and describe two possible student perspectives of the concept giving rise to a conceptual process representation tool (CPRT) or a conceptual object representation tool (CORT). Janvier (1998) has also discussed how students’ discrete, point-wise interpretation of function graphs is deeply anchored in students’ schemas, preventing them from moving to more powerful global perspectives. According to Thomas & Hong (2001) it will also constrain the interaction they can have with a representation. Flexibility in interactions with quadratic function representations was the focus of Even’s (1998) study, but she found that students could not easily see a quadratic expression in terms of its graphical representation.

One of the factors influencing a learner’s interaction with a representation will be his/her cognitive structures associated with the conceptual ideas it represents. An important area where this has been demonstrated is with regard to the process/object distinction of mathematical conceptions (Dubinsky, 1991; Tall et al., 2000). For example, a point-wise view of a graph arises from a process perspective of function, whereas the global view requires one to see the graph as representing an encapsulated object.

Kaput (1989) has long suggested that technology such as GCs with their multiple linked dynamic representations could support the building of new meanings across algebra. In agreement with this, Ruthven (1990) has demonstrated that when using GCs to link graphic and symbolic representations students were able to recognise that a given graph came from a family of curves and the use of multiple-representational features of the GC enriched students’ problem solving strategies. Similarly, a study by Asp, Dowsey, and Stacey (1993) with year 10 students looked at the influence of GCs on quadratic function graphing and reported significant improvement in interpreting graphs and matching graph shape with symbolic algebra forms. The study by Harskamp, Shure, & Van Streun (2000) on the effects of GC use on problem solving performance in the domain of functions and calculus showed that access to GCs increased the use of students’ graphing strategies without reducing other strategies, but only after a prolonged period of at least 1 year.

It appears that the opportunity exists to use GCs to encourage cognitive links by examining the quadratic function in different representational contexts. A primary aim of the current study was to address this and describe student interactions with different representations of quadratic functions, and whether they could build representational flexibility and fluency in this way.

Method

A group of twenty-five male and female year 10 students (age 14~15 years) from a second stream class at a private school in Auckland took part in this research, although not all completed all the tests. In New Zealand, students are first introduced to symbolic manipulation of a quadratic expression in year 10 and this includes factorising quadratic functions and solving quadratic equations. However, it is not usually until year 11 that most students are introduced to the graphical representation of a quadratic function. Although the school involved in the project encourages the use of technology, and in particular every student has their own laptop computer, none of the students had ever used a graphic calculator in mathematics before.

Instruments

Each student was supplied with a resource booklet containing a teaching module which integrated use of the symbolic, tabular and graphical representations. The booklet included step-by-step operating instructions together with calculator keystrokes and screen dumps for the different calculator modes used (see Figure 1). In addition, parallel pre- and post-tests were constructed (with the post-test also used as a delayed post-test) in an attempt to gauge students’ equations solving skills (including quadratic equations in a standard symbolic representation usually seen in school mathematics text books) and their conceptual understanding of finding solutions to quadratic equations both within and between representations. Some might consider using the same test for the post– and delayed post–tests problematic because students have learned from the test, but to counter this students were not given their tests back prior to the completion of all testing. The students were able to use scientific calculators in the pre– and post–tests, but the GCs were only allowed in the delayed post–test.

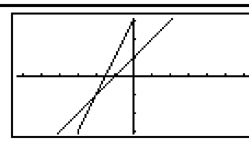
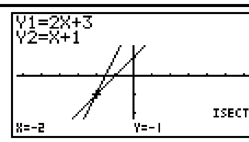
	Keys	See	Things to think about
Change the viewing window settings to (INIT) initial setting.	F1 (INIT) EXIT		
Draw the graphs of the functions $y1$ and $y2$	F6 *(DRAW)*		
Select the GRAPH SOLVE menu options	SHIFT F5 (G-Solv)		
Find the point of intersection (ISECT) of the pair of graphs.	F5 (ISCT)		The solution of the equation $2x+3=x+1$, will occur when $y1=y2$ i.e. at the point of intersection of the two graphs. Write this as a co-ordinate pair (,).

Figure 1. *An example of the teaching module's inter-representational GC operating instructions.*

Procedure

We considered it essential to involve the class teacher in the research process and so one of the researchers met with him a fortnight before the implementation phase. He was given a copy of the suggested resource material and the objectives of each of the lessons were discussed, as well as how he could be involved in the teaching. A class set of Casio CFX-9850G GCs and a calculator viewscreen were used in the research. Unfortunately, due to the nature of the schools' insurance cover, it was not possible to allow students to take the calculators home. This was a major disappointment, since it was expected that it would have generated increased interest in, and familiarity with, the GC.

In the four lessons immediately prior to the research period the students received instruction, without GCs, on expanding and factorising quadratic expressions, solving quadratic equations and graphing functions of the form $y=x^2$, $y=x^2-1$, $y=(x-2)^2$, and $y=(x-2)(x+3)$ by plotting points. In addition, since the students were unfamiliar with the GC, they had one familiarisation lesson during which they were given some examples of the type of mathematical problems they were likely to encounter in the module, and had the format explained (see Figure 1).

The teaching module of three fifty minute lessons was very focused, introducing students to the multiple-representation capabilities of the calculator, including the Dynamic mode (initially by generating patterns), and using these to solve quadratic equations via a flexible, inter–representational and conceptual approach. Each student was given an individual booklet containing the instructions and worksheets for each lesson (but not the homework worksheets) in which there were spaces for them to write their work (see Figure 2). The students were organised into small groups to encourage discussion and co–operative learning, but waited for instruction and guidance from the teacher before progressing to the next section. The class teacher and at least one researcher were present at all times and when not involved in teaching the class they circulated amongst the groups to observe, answer questions and assist with difficulties. In the period following the post-test, and before the delayed post-test (approximately a fortnight after the post-test), the teacher was asked to continue to make a calculator available to each student. During this time the students had two further lessons on solving quadratic equations and a school test.

The weekly wage (call this $W) of a vacuum cleaner salesperson consists of a fixed sum of $350 plus $20 for each vacuum cleaner sold. Let V represent the number of cleaners sold per week. Complete the ***table*** *below to show how the weekly wage depends on the number of cleaners sold.*

V (Number of cleaners sold)	**W** (Weekly wage)	Co-ordinates
0	350	(0,350)
1	370	
2		
3		
	490	

Construct a **rule** which describes the weekly wage of the salesperson:

Weekly wage = ☐ Vacuum cleaners + ☐

W = ☐ **V** + ☐

On the axes below, plot a set of points to show the relationship between **W** and **V.** Join these points.

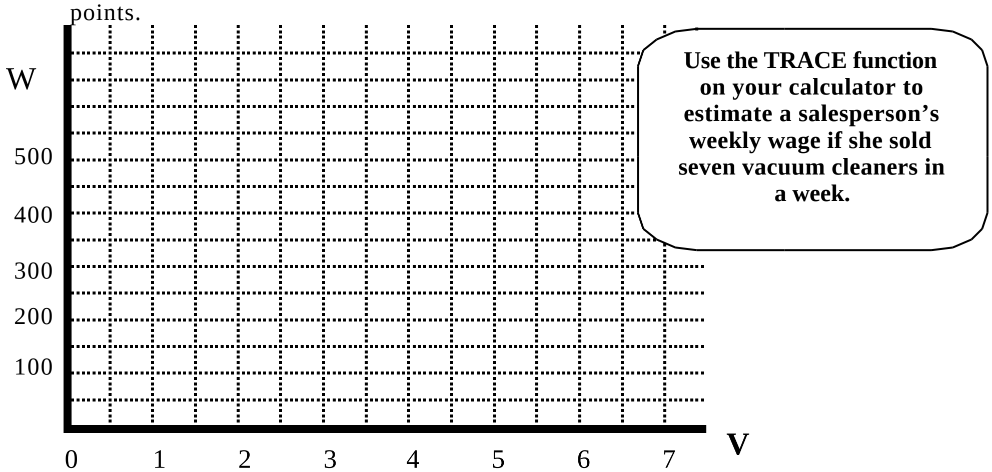


Figure 2. *A section of the teaching module showing the layout and use of three representations.*

Results

The questions on the tests were designed to help answer several questions about students' representational interactions, skills and understanding, such as:

- What was their level of ability to work within the symbolic representation, including understanding of conservation of equation under addition of $mx + c$ (one of m, c equal to zero)?
- Could they solve an equation presented in one representation by using another?
- Could they relate processes for solving equations in tabular, symbolic and graphical representations?

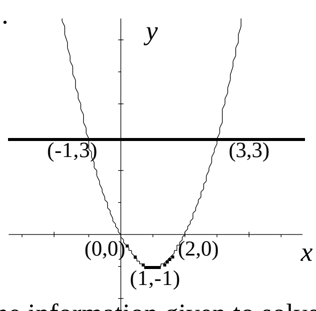
Certainly, overall, the students did better on the test after the short module than they did before (N=17, pre-test mean=4.09, post-test mean=6.12, t=2.74, $p<0.01$), and there was some evidence that this improvement was sustained through the delayed post-test (N=17, pre-test mean=4.09, delayed post-test mean=5.38, t=1.81, $p<0.05$). However we were more interested in where the improvements had occurred, the relationship to the representations, and what kinds of difficulties the students had and why.

There was no change at all on the solution rate for the five symbolically presented quadratic equations (pre-test mean=1.04, post-test mean=1.09, t=0.25, n.s.), such as solving $(x+7)(x+5)=0$; $2x^2 - 12x + 18 = 0$; or $p^2 + 6 = 5p$, or on the five conservation of equation questions which asked whether pairs of equations such as $x^2 + 4x = 12$ and $x^2 + 4x + 4 = 16$ or $s^2 - 3s = s + 1$ and $s^2 + s = 5s + 1$ have the same solutions (pre-test mean=1.76, post-test mean=1.78, t= 0.11, n.s.). The lack of progress in procedural skills in the symbolic algebra representation was not unexpected since it was not a major consideration of the research, but was included as a benchmark. The relatively poor performance on conservation of equation under addition of a constant or multiple of the variable indicates that students do not understand the principle underlying the balancing method of solving equations in the symbolic algebra representation. If they use this method then they will be applying a procedure without understanding the basis for it.

Questions Involving Inter-Representational Linking

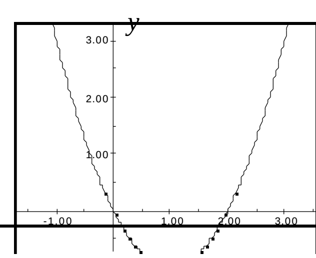
The test also contained questions where a quadratic equation was presented symbolically but was required to be solved in a graphical or tabular representation. In Figure 3 the left hand question requires understanding that the symbolic functions can be represented by the graphs and that the solution of the symbolic equations can be found at the intersection of the graphs or the x-intercepts. The second question is a little harder since students do not simply have to match the graphs to the symbols but have to draw in an appropriate graph first. They have to link the two representations and then follow with an appropriate process within the graphical representation.

The two graphs drawn below are $y = x^2 - kx$ and $y = 3$.



Use the information given to solve the following

The graph of $y = x^2 - 2x$ is drawn below.
Use this graph, and any other lines which you may need to add to it, to solve the equations given. Briefly explain your method.



equations for x: 1. $x^2 - kx = 3$ 2. $x^2 - kx = 0$	1. $x^2 - 2x = 3$ 2. $x^2 - 2x = x$

Figure 3. *Questions involving linking of symbolic and graphical representations.*

Many students were unable to make connections between the different representations, and in particular could not see how graphs could be used to solve equations in this way. Even though the question had specifically asked that a graphical solution be found to the equation $x^2 - 2x = 3$ a reliance on the symbolic form of the equation and often a pointwise process perspective of function, seemed to lock some students into symbolic mode and the thinking associated with it. This in turn affected their interaction with the graphical representation. One student, B1, responded by describing her trial and error input-output procedural method this way: "The graphs really don't contribute at all to my answer, in algebra. I just try to find *x* with numbers in my head and not numbers on a graph". The students, S1 and H2, whose post-test solutions are shown in Figure 4, have used the graph to draw up a table of points and then used this to solve the equation, providing evidence of their pointwise, discrete view of function rather than a holistic, object view. While these students are using the graph as a *conceptual representation tool* to solve the equation, mediated by a third representation, the table, it is based on a process view of function.

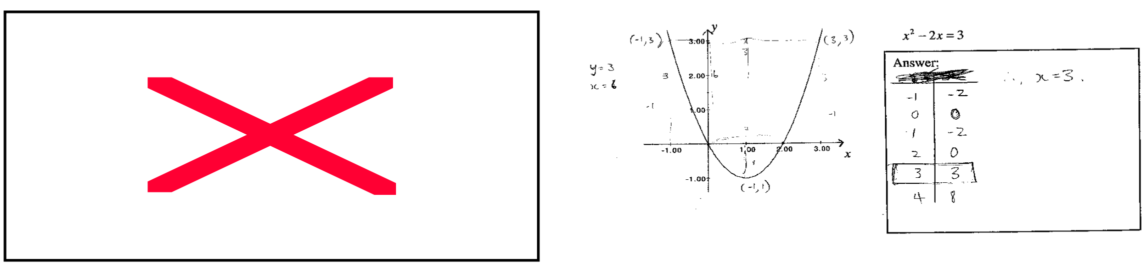


Figure 4. *Students S1 and H2's use of a discrete, pointwise view of a graph to solve an equation.*

In the post-tests other students were able to interact with the graphs as a conceptual representation tool, relating the symbolic equation to a graphical function object, by drawing a straight line and finding out where the two graphical objects intersected (see Figure 5). They gave no evidence of needing a pointwise view of the function in its graphical representation, but often gave only a single, positive solution.

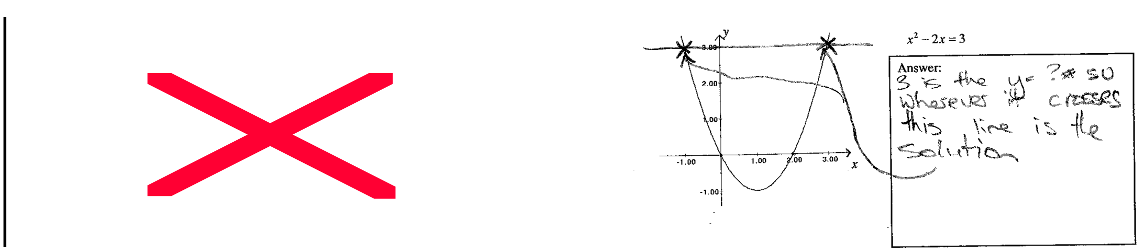


Figure 5. *Students K2 and S4's use of an object view of a graph to solve an equation.*

These students, K2 and S4, were among a group of five whose scores on the delayed post–test were higher than on the post–test, and four marks on average higher than the pre-test (maximum mark=31). This would suggest that they required time to process new ideas and concepts and to link these to their previous knowledge and to other domains. This progress is seen in S4's solutions to the question in Figure 3, as shown in Figure 6. To begin with, she interacts with the symbolic representation procedurally, attempting to rearrange the equation with k as a constant. In the post-test, a procedural guess and check technique, still embedded in the symbolic representation, is used to solve the equation for a specific value $k=2$, with no indication of how this value for k is obtained. However, in the delayed post-test the symbolic and graphical representations are related and she interacts with the graph conceptually to read both solutions directly from it.

Generally speaking, students displayed a poor understanding of the use of tables, no one solving equations using a table which required them to perform an arithmetic operation such as 'subtracting three' from both sides before it could be used to solve the equation.

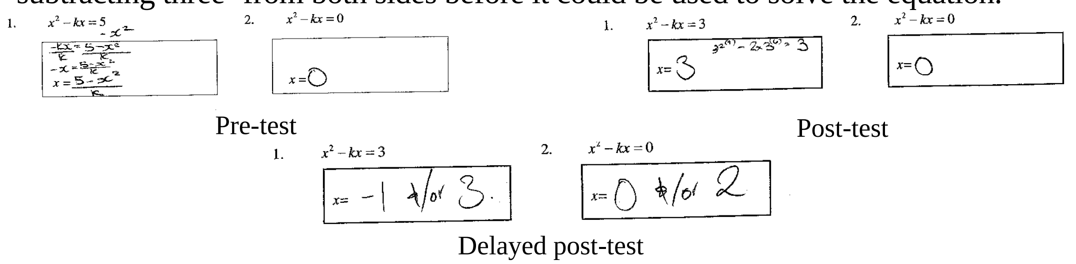


Figure 6. The progressive interactions of student S4.

Most interacted procedurally with the tabular representation, either substituting each of the given values of x in the left hand column of the table into the equation (and failing to find a solution), or finding the value on the right hand side of the equation in the y column of the table, and giving the corresponding x value(s) as the solution. Figure 7 records typical interactions with the tables, here of student S1, in the three tests. In the pre-test, he attempted an algebraic solution, without success. His next attempt indicates that he initially thinks that –6 and –2 could be the solutions as these values for x correspond to 4 in the y column. However, algebraic substitution verifies that this is not the case. Realising that the solution will be a value of x for which $(x+4)^2$ must be a less than 4, he employs a procedural, pointwise guess and check technique, working within the symbolic representation but linked with data from the tabular one, establishing that –5 and –3 are the correct solutions. In the delayed post-test he again gives x=[-]6 and x=–2, the values which correspond to y=4, rather than y=1.

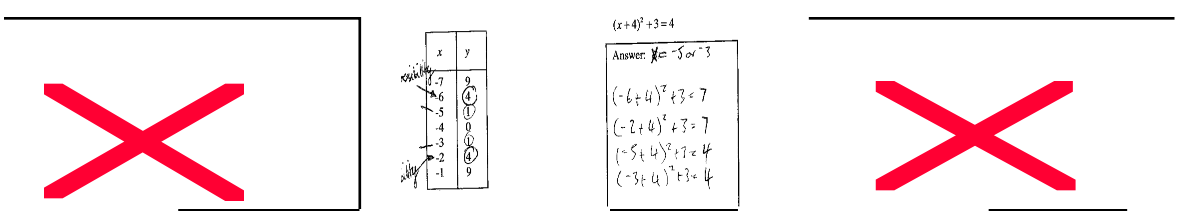


Figure 7. Examples of process interactions with tabular representations.

Conclusion

Lesh (2000, p. 74) suggests that "… representational fluency is at the heart of what it means to 'understand' many of the more important underlying mathematical constructs" and we can see that much is involved in gaining such representational fluency. It includes the ability to interact with these representations, using them as conceptual tools and to demonstrate the flexibility of being able to move from one representation to another, recognising invariant properties, etc. The results of this study show that the use of the GC was not as successful as we had hoped in building such fluency. Neither did the students improve at all in their solutions to quadratic equations (as expected) or with their understanding of conservation of equation under addition of one of *mx* or *c*.

The students' inter–representational abilities followed a similar pattern. They were unable to use a table to solve equations presented symbolically when the table did not correspond directly to the function in the algebraic form. They also struggled, with a few exceptions, to relate the algebraic symbolism to relevant graphs which they should have been able to use to solve equations. One of the key reasons for their difficulty is their point-wise view of function as a relation between two sets; a value for value perspective that they carried through each of the algebraic, tabular, co–ordinates and graphical representations. Their interactions with each representation were primarily process-oriented. Awareness, as Penglase & Arnold (1996) point out, of the difficulty of separating GC and pedagogical effects when teaching with GCs is necessary, and there were a number of contributory factors which exacerbated the situation here. These included the inability of the students to take the GCs home; several disruptions to the GC lessons due to school activities; and the relatively short time the students were able to spend on the work.

What is clear from our research is that students need some classroom experiences to assist them to construct both process and object conceptions of function across a number of representations. While we were not successful in providing the kind of inter–representational work with the GCs which addressed their problems, we would not be quick to say that the GC is not a useful tool. We remain convinced that, in theory, the GCs have the potential to help students develop a wider conception of function transcending representation, and including aspects delineated by Janvier (1998), and that effort should be put into finding a suitable pedagogical format to deliver this.

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