Prospective Teachers’ Perspectives on Function Representations

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This is a report of a study in which we aim to characterise the quality of subject-matter and pedagogical content knowledge constructed by a group of four student teachers in the domain of functions. A key feature of the results is that these teachers tended to emphasise the visual representations in their understanding of functions without sufficient attention to their symbolic equivalency. As expected, their pedagogical content knowledge was not well developed. We discuss the implications for subject-matter growth from the framework of modelling and schema development.

The study of functions constitutes an important requirement at many levels of schooling. Early developments in the understanding of functions involve students’ ability at modelling one-to-one correspondences between members of a given set of objects. Such a relationship can also be given an algebraic interpretation as students advance to higher levels of understandings. Students with deeper understanding of functions can also be expected to articulate the many- or one-to-one correspondences that underlie the algebraic expression of a function. Thus functions can be represented in a wide variety of different ways, such as in algebraic symbol form, set diagrams, ordered pairs, graphs and tables of values.

The multidimensional nature of functions presents particular challenge to the teachers in designing appropriate learning situations. Translating between these various forms of function representations is something which an experienced teacher might take for granted, but to do so one needs to have an overview of the way the definition of function relates to each representation, and how sub-concepts, such as independent and dependent variables, one-to-one, roots, discrete, continuous, etc. are manifest in each representation (Hong, Thomas & Kwon, 2000).

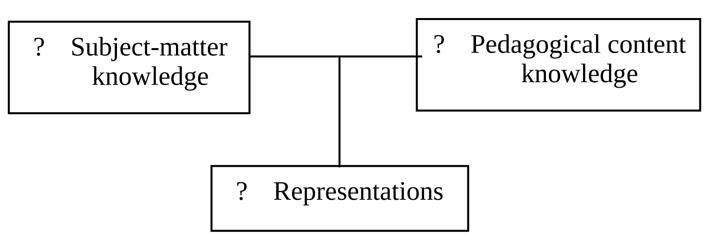


Figure 1. Representations mediating use of content knowledge in teaching.

Among others, teachers draw on two major components of their knowledge base in planning and delivering effective lessons on functions: subject-matter knowledge (Ball and McDiarmid, 1990), pedagogical content knowledge (Shulman, 1986). Included in the teachers’ repertoires of subject-matter knowledge are substantive mathematical knowledge such as facts, ideas, theorems, mathematical explanations, concepts, processes. The pedagogical content knowledge includes information that the teachers hold about the students’ understanding, such as their conceptual and procedural knowledge about functions and misconceptions. Thus a competent teacher would not only have built a rich store of these two knowledge components, but also have integrated that knowledge so that he or she could activate subject-matter knowledge appropriate for the students. We argue that this integration involves the development of a greater variety of representations for functions. That is, teachers who have constructed more varied representation of functions will not only have a richer subject-matter knowledge but also able to translate and use this knowledge in ways that would help students assimilate that knowledge (Figure 1).

In our view technological tools can assist in the building of links between the sub-concepts in each representation. The key property they possess is the fact that the representations can be dynamically linked, with the technology providing fast feedback on student input (Kaput, 1992), so that a change in one representation, such as a table, is immediately mirrored by a corresponding change in the graphical representation. Figure 2 shows the potential of technology in promoting the construction of links among four key representations of function (Chinnappan and Thomas, 2000).

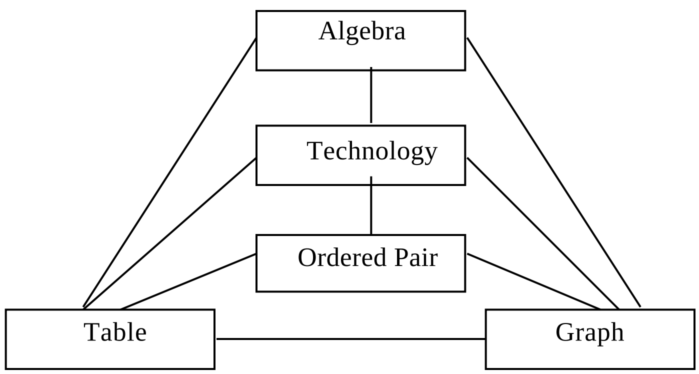


Figure 2. Technology as a catalyst in the linking of function representations.

Method

This research employed a case study methodology, considering the function schemas of four first year full-time graduate teacher trainees, A, D, M, and V, who had only taught mathematics on practicum. The teachers were given a free response interview in which they were encouraged to talk about ‘functions’ and how they would teach them, and these were recorded on audiotape and later transcribed for analysis. Our intention is to draw a detailed contrast between the schemas of these students and the experienced teachers we had previously interviewed, but this preliminary paper describes only the schematic thinking of the trainee teachers.

Results

In our analysis of the data we were interested in identifying these prospective teachers’ knowledge about functions, and the use of technology in fostering the development of such knowledge, as well as seeking to relate this to the macro model of the structure of teacher’s mathematical knowledge we have been using for the analysis of conceptual understanding developed by teachers (Chinnappan & Thomas, 2000). This model stresses that teachers of mathematics need to have a broad view of mathematics and its learning. They should not be limited to seeing it as primarily a skills-based, algorithmic subject, nor should they be constrained to thinking in terms of a single representation when dealing with any given content. Rather, teaching mathematics should be seen as involving the student construction of concepts and the network of links between them and their sub-concepts (Vollrath, 1994) across representations.

Representational Thinking

One of the things that the student teachers did in our interviews was to talk about the definition of function as they saw it.

> A: The two things I think that I understand when I talk about functions, is that spring to mind immediately out of the vertical line test because that's what I learnt. OK so I know that to find the function you apply the vertical line test and you apply that because um…you want the function to have only one *x* value… If you've got a function like that, you want it to cut the graph only once and therefore only getting one *y* value so that's.. When I think of that I think tests. Immediately I think vertical line te.st

> D: To me uh.. definition of a function is a, maybe a special type of relation uh.. I understand that functions come under the heading of um.. relations. Um.. but uh.. I would say that we only get for a function, um.. we only get one value for the.. say an *x*, in a graph situation, for an *x* versus *y* we only get one. … variable. Do you want me to draw a little graph?… the algebra is the actual expression or working with the actual expression. And the function to me is the visualization of that graph . How is that?

> M: I got the idea that functions was a graph. Um.. it was a little curve or a straight line on a graph. …I think it was in fifth or sixth form that they defined what function was and um they had this the vertical line test where they said OK, given for example *y* equals for example, *x* cubed… if you were to draw that on a graph by using the vertical line test and if it cuts the particular graph at one point on the *x* axis if I've got this right, and then that is what you call a function.

One feature was immediately striking about these responses, and this was to dominate all the students teachers' interview comments. The student teachers had a strong tendency to think of functions graphically and in terms of process. A, M and V all specifically mentioned the process of the vertical line test on a graph, and D actually drew the test in a sketch. Not only do they think of functions graphically but to M this relationship goes further. She thinks of them as actually being graphs, drawing a clear distinction in her mind between functions and algebra: "Yes it's less time consuming and so they'll be able to concentrate on what the graph looks like, a function really looks like, rather than calculating. … I think that what's really important is what they see what the actual function looks like rather than spend more time on algebra."

As one would expect, the richness of these student teachers' function schemas varied somewhat, with some displaying understanding of a large number of related and sub-concepts while others were much more limited. However, what was noticeable was the extent to which understanding of these concepts was often mediated by a graphical or other visual representation for two of the teachers, A and M. Student teacher A, for example, in her interview has a clearly expressed preference for 'pictures' in mathematics, and only associates stationary values, even and inverse functions with graphs, and links this to her school experiences.

> A: To find where the function will be at a maximum value um… and minimum values. I guess the stationary points um.. points of inflection, all interesting features of the graph and I'm pretty short

of examples

Once you understand the sine and cos and tan curves once that comes to you it's so nice.. it's a security thing, when all else fails you can go back to those and that's just fantastic to know that. And when you get completely stuck draw a little sketch and then something will come to you then you'll think uh

Similarly, student teacher M prefers to view the sub-concepts of increasing and decreasing functions, rates of change, roots, limits, maximum and turning points in terms of gradients and tangents, etc of graphs, "rather than the algebra itself."

M: whether it's increasing or decreasing all those other kind of terms that come in um.. uh.. what they say, the rate of change or you know the gradients, that kind, so they'll be able to look at that rather than the algebra itself. There is also the idea of limits for example, you can select two particular points on a graph on the function and as the two points get closer and closer together they form a gradient…

Oh, just drawing my x and y axis now.. and I'm just doing any particular graph.. here. And then you see that these are the roots and also got the point of the maximum and I think that's important that students know that at that particular point that the gradient.. the gradient is zero and even that's our minimum, so differentiation comes in.. uh, there is also um.. so therefore turning points.

Subconcepts and Representational Links

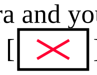
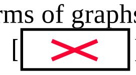
At times, probably due to their overemphasis of the graphical domain, these teacher trainees seemed to lack the ability to relate some concepts across representational boundaries. For example when discussing composite functions A wanted to be able to understand what they mean in the graphical domain but was unable to, and said "I know how you do it but I don't believe I could tell you why you do it or what the graph would look like, if you had for example, x squared minus nine is your f function, um… and your g of x [$g(x)$] was a three x plus two. How they look on a graph is completely beyond me for a start. I wouldn't be able to tell you what they look like."

Sometimes a change from one symbolic representation to another can present cognitive obstacles, as illustrated by A's remarks about the change from $f'(x)$ to :

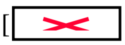
A: I couldn't do  and as soon as I hit university, it changed and I don't know what made me change from $f'(x)$, because through school, sixth and seventh form, $f'(x)$ was, I mean, that's what I worked with. I liked that. And then all of sudden it became necessary to use this form although I got completely stumped in first year calculus with it. And it took me a long time to figure it out, but once I did it was great, like a revelation.

This seems to indicate a lack of understanding of the concept underlying the notation suggesting the learning of $f'(x)$, which occurred in the comfortable school environment, may have been primarily procedural and anchored in the symbolic representation.

There were other general comments about the desire for linking algebra and graphs, such as that of M.

M: …maybe adding functions would be the same as adding the polynomials, and you can relate that to adding functions as part of algebra and you can present adding functions in terms of graphs, for example, graph y equals x squared [] and y equals x squared plus one []. Adding those two functions they see how it relates visually on graph. And then also um.. in algebrawise, what you come out with. So the outcome in algebra and the outcome with graphing it.

However, it is manifest that she does not have a clear conceptual view of functions here, distinguishing a dichotomy with polynomials being in the algebraic domain and functions being graphical, as we have already seen above. For V the graphical representation in his mind is an obstacle to understanding, preventing him from separating out the independent variable in the algebraic form. He spoke of how "say we've got the relation being a circle, yet we can perform an operation on that circle say sine of the *x* value. And that comes up a function. You can convert the relation into a function and I found that unusual." D on the other hand was able to relate some ideas across representations. He spoke of the link between solving a quadratic equation by factorisation and by using the graphical intercepts, and the value of this for students.

> D: *x* squared plus three *x* plus two [] is our expanded form. O.K. so on a graph, um.. our intercepts can be found by this factorised form of that.. obviously I've first of all started with a factorised form and then expanded it so that it was easy for me to come up with it. But um… when we're looking for a cutting point on the *x* axis, these factorised forms.. so *x* equals minus two will give us *y* value of zero so we'll have a minus two and minus one…. And then.. so therefore I think that the graphing solution is a good example of why the factorising is so important so if you can show the two things at once for a student it is a good link in your mind.

The trainee teacher D seemed to be much further along in the process of constructing his pedagogical content knowledge than the other three students. This is a good example of the way he is thinking about the advantages for the students he will teach.

Limited understanding

The trainee teachers recognised that they had gaps in their knowledge of function, and some function sub-concepts were not well understood, or not linked to other concepts. They also held some ideas which are incorrect. Talking about composition of functions, $f(g(x))$, A struggles to remember what it is, and the procedure she learned for calculating it.

> A: I'm not completely familiar with off the top of my head I do have to look at them and say o.k. well, the domain becomes range for the.. if it's this way around, and vice versa, the range becomes domain if it's that way around, I don't know that off the top of my head…. I know how you do it but I don't believe I could tell you why you do it…I would have to probably look up which way around it goes, um…from memory, it's backwards. So I put the *f* into the *g* I think. *f* of *g* [$f(g)$]. Um…just thinking about that. I would have to check that. I'm not a hundred percent sure of that. But then I can get *g* of *f* [$g(f)$] so I presume there's something special about which way around it goes um, I'm not entirely certain of why or which way around.

It would appear that procedural learning has contributed to this problem. Her expressions indicate that she learned this topic instrumentally, without relational understanding. If she had a relational view of the symbolic notation then she would 'see' that in $f(g(x))$, $g(x)$ has to be found first, and then *f* is applied to the result, so she wouldn't be saying "So I put the *f* into the *g* I think". A also has hazy knowledge in the area of the fundamental theorem of the calculus. Having associated integration with areas under curves in her schemas she has great difficulty defining indefinite integration.

> A: Indefinite, you'll get a function which is.. you'll get another function or constant, no, rewrite this, I'm not sure about that. Um.. and an indefinite function will give you another function, that if you, sorry, are we integrating? If you integrate a function you get another function that will allow you to calculate the area, I'm not a hundred percent sure on that, … oh gosh it's confusing.. you get another function plus a constant and what does that do for you.. it allows you to calculate area when we integrate, oh.. .. [long pause] I think I'd have to read that.

Student teacher M has difficulty with points of inflection, a concept which she admits she has not understood "There is also the point of inflection which they say, it's,.. don't really understand.. but what I can.. it's where …um…it's ….That I am quite stuck on that point of inflection. … It's quite flat…I think.. I might say something wrong." V has a problem with the concept of an odd function, having failed to make the link with polynomials of odd powers he invents a possible reason for the name 'odd':

> V: To me there is nothing there that relates to what odd numbers are. … To me like, odd function it doesn't have any relation to odd numbers. I mean if I was to try and think about it, I would think this one here is an odd function because it's got an odd number of minus negative signs and the other one's an even function because it's got even number of negative signs so if I've got them round the right way, they don't fall into any logical pattern.

One of the subconcepts of function which two of the teachers had difficulty with was a polynomial. V for instance finds it hard to relate them to functions, and does not consider a linear relation as a polynomial:

> V: If somebody said 'is that straight line relation a polynomial?', my gut reaction would be to say no. Just because a polynomial poly being many. … O.K, the straight line graph comes under the linear heading all the orders above one come under the polynomial heading… I find that when I am talking about functions, I am not talking about polynomials and vice versa, I find it very difficult to um.. interchange the words.

Student teacher A too admits to problems with polynomials, "I did have to look up the definition of this when I started teaching because I wasn't a hundred percent sure on what it was. … I use always quadratic polynomial and sometimes I don't know what they mean. I've forgotten what they mean. I don't know whether I ever knew that anyway."

Applications

One of the strengths of experienced teachers who we had previously interviewed (Chinnappan & Thomas, 1999), is their ability to think and structure the mathematics they teach in terms of applications and modelling. Thus their lessons often comprise the gathering of realistic data and then using this to model a practical situation. These student teachers found the idea of applications of functions difficult to describe, being limited to the ones they were taught at school, which are primarily in the calculus, and often in the context of growth and decay functions described using differential equations. M talks of these, saying "I can look at things like um…population growth where I think that might go a little bit overboard 'cos that'll look at exponentials." Two of the others comment:

> A: To be honest my main experience of functions has been in the classroom. I haven't had any real world experience with the functions… I'm sure that there are many, many applications but just to give you just a few examples of, I suppose let's take a weta [NZ insect] population, that's dependant on many factors and when it levels out, or when some of those factors will influence the population numbers, either greater or less than, and possibly interested in that seasonal and all that sort of stuff so you could possibly do a [unclear adjective] function for that.

> V: To me, when I ask or think about applications of functions and polynomials, like where in everyday life can we use them to model situations things like that. Um.. like say we've got.. like. Like.. uh.. I think we, at Bursary level when we had the um.. differential equations the growth curves and things like that, bacteria. I think it's that area of the maths was good because it gave you practical applications for the underlying maths.

In terms of our model of teachers' mathematical knowledge in (see Chinnappan and Thomas, 2000) these teachers will have to work hard to bridge the gap between their conceptions and the idea of mathematical modelling, since their knowledge of applications is very limited in scope.

Technology Use

All four of the teachers expressed a strong liking for technology in mathematics learning, especially computers and graphics calculators, but because of their graphical preference in the context of function they apparently see it principally as a graphical aid, speeding up the drawing, or as A says it "takes all the tediousness out of drawing another graph" or as providing conceptual benefits but through its visual power.

> D: I think it's hugely valuable for them to see how the graphs.. how the computer works and they can see what's happening and they can.. because I think a lot of students have difficulty with a concept of a limit.. …I've seen the computer visually actually doing and how that value of that gradient comes down to a set value is really useful for the students.

> V: I think that using calculators and computers is becoming more and more important. Because not only are they quicker and easier giving you a display of what you've got in terms of your function but you can play with the function. … with graphics calculators and computers, it's easy to draw … and it's easy to change the different parts of the functions …and you can straight away see how it affects what the graph looks like rather than having to draw it out and plot all the points.

However, the teachers have a limited view of the value of technology as a teaching tool. While D is relating limit to the graphs he is not yet linking the symbolic forms to the pictures. They do not really see its capability as an inter–representational tool, the possibility of using it as a catalyst for promoting links between various domains, as described in Figure 2. In order to make best use of the computer or calculator the teachers will need to integrate the technology into their function schemas in such a way that these links are made. Most of them will need assistance to do this.

Discussion

The results show that the four trainee teachers we interviewed all tended to relate their function content knowledge to what they had been taught at school, from where, for example, the vertical line test emanated. While one might superficially expect this, their later tertiary studies appear to have had much less influence on deepening this understanding. For example, M specifically states that "it was in fifth or sixth form", and even V who begins with talking about tertiary studies where "you get told that functions are a special type of relation", is not totally happy with this and quickly reverts to what he was told at high school, deciding that "if I am asked what is a function I'll say any graph on the x- y plane that… you can draw a vertical line down." This demonstrates how influential secondary school learning is even for successful mathematics students. The teachers thinking was structured in their minds by how they had been taught and school and they needed to make a switch to thinking about how they will teach it.

A key aspect of the student teachers' responses was their fixation with visual representation of functions with little concern about the links to their symbolic equivalents, and the potential effect for student learning. This finding is consistent with recent studies on functions which have shown that students tended to experience difficulty in interpreting graphical and symbolic representation of functions (Leinhardt, Zaslavsky and Stein, 1990; Mitchelmore and Cavanagh, 2000). We argue that the continued emphasis on visualising functions without drawing out the underlying algebra will do little to change the above deficiency in students' understanding of functions.

In spite of the dominance of school mathematics, the student teachers we have interviewed do seem to be engaged in the initial stages of the process of accommodation of their function schema as they prepare to teach mathematics in school. As they examine their content knowledge and begin to construct pedagogical content knowledge of function, they are coming to recognise, as they did in the interview, that they have gaps in their function schema, but are confident that they can fill these gaps once they are teaching. One way that they have already begun this process of development is by learning from colleagues in the schools where they are on teaching practicum. It is clear that this is a two-fold process where they are learning ways to teach function (pedagogical content knowledge) and increasing their subject-matter knowledge by increasing the network of information embedded in their function schema. Student teacher D referred to two specific areas where he has benefited from his observation of an experienced teacher: the linking of graphical and algebraic representations of functions; and the utilisation of computers in order to construct these links in a dynamic manner.

One lesson arising from the above analysis is that when they begin in their first teaching post student teachers have a continuing need to be supported by their colleagues. Experienced teachers should take the time to talk to new teachers about how they might present concepts such as function to their classes. Three areas in particular where this study suggests that this would be particularly advantageous with regard to function are: a) inter-representational approaches to the concept of function, b) use of computers and graphic calculators to promote representational links for functional concepts and c) ideas for realistic examples for using modelling of functions.

References

Ball, D.L. & McDiarmid, G.W. (1990). The subject-matter preparation of teachers. In Houston, W.R. (Ed.) *Handbook of Research on Teacher Education.* New York: McMillan

Chinnappan, M. & Thomas, M. O. J. (1999). Structured knowledge and conceptual modelling of functions by an experienced teacher, *Proceedings of the 22nd Mathematics Education Research Group of Australasia Conference*, Adelaide, 159–166.

Chinnappan, M., & Thomas, M. O. J. (2000). Function representations and technology-based learning. In J. Bana and A. Chapman (Eds.), *Proceedings of the 23rd Conference of the Mathematics Education Research Group of Australasia* (pp. 172-180), Fremantle: Mathematics Education Research Group of Australasia.

Hong, Y. Y, Thomas, M. O. J. & Kwon, O. (2000). Understanding Linear Algebraic Equations via Super–calculator Representations, *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, Hiroshima, Japan, 3, 57–64.

Kaput, J. (1992). Technology and Mathematics Education, *NCTM Yearbook on Mathematics Education*, 515-556.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs and graphing: Tasks, learning and teaching. *Review of Educational Research, 60*, 1-64.

Mitchelmore, M., & Cavanagh, M. (2000). Student difficulties in operating a graphics calculator. *Mathematics Educational Research Journal, 12*(3), 254-268.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching, *Educational Researcher, 15*, 4-14.

Vollrath, H-J. (1994). Reflections on mathematical concepts as starting points for didactical thinking, In R. Biehler, W. Scholz, R. Straßer, B. Winkelmann (Eds.) *Didactics of Mathematics as a Scientific Discipline,* (pp. 61-72), Dordrecht: Kluwer Academic Publishers.