The Multifaceted Variable Approach: Selection of Method in Solving Simple Linear Equations

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This paper presents a comparison of the solution strategies used by two groups of Year 8 students as they solved linear equations. The experimental group studied algebra following a multifaceted variable approach, while the comparison group used a traditional approach. Students in the experimental group employed different solution strategies, namely balancing method, working backwards and guess and check for solving different linear equations, whereas students in the comparison group tended to use a single, procedural approach. It is concluded that the multifaceted approach developed students' concepts not only of variables but also of equations.

Students' understanding of core algebraic concepts of variable and equivalence influences their success in solving problems, the strategies they use, and the justification they give for their solutions (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). Students who interpret letters as specific unknowns and not as generalised numbers or variable quantities learn the procedures of manipulation and substitution without assigning any meaning to the symbols (Booth, 1995). Misconceptions about the variable, such as "whenever a letter stands alone it is equal to 1" and "letters and numbers are detached", are also responsible for student difficulties in equation solving (Perso, 1991). These misconceptions are carried forward into concepts like equality and equation solving and students think that the meaning of the equality sign is always an instruction to find the answer by carrying out some calculation (Kieran, 1981). Students misapply rules for transforming equations, which could be due to misinterpretation of algebraic expressions or not understanding the given situation (Nunes, Bryant, & Watson, 2007). The results of the National Assessment of Educational Progress indicated that mathematics instruction in Year 8 focused on learning skills and procedures rather then developing reasoning ability or communicating ideas effectively (Mitchell, Hawkins, Jakwerth, Stancavage, & Dossey, 1999).

Different teaching approaches such as a functional approach, a problem solving approach and a generalisation approach have been suggested from time to time as a way of teaching beginning algebra. However, research has also indicated a range of difficulties associated with each of these approaches (Booth, 1988; Küchemann, 1981; Sfard & Linchevski, 1994). Trigueros and Ursini (2001) presented a teaching model to approach the study of algebra. In this 3UV model, they suggested that three aspects of variables — namely unknown, variable and generalised number — should be studied one by one and then these three aspects should be integrated so that students can acquire a holistic concept of variable. The effectiveness of this teaching model is not supported by any reported research. The fact is that the aspect of a variable as an unknown quantity is automatically included in the other two aspects (variable as a function and variable as a generalised number). However, generalised numbers associate variables with multiple values and are not sufficient to indicate the relationship between quantities, which is an essential ingredient in representing a problem algebraically. Therefore it is necessary for variable as a function to be studied together in parallel with generalised number using multiple

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representations and real contexts so that a complete meaning can be associated with the term ‘variable’. After studying variables in Year 7 students can then move on to symbol manipulations and the solution of linear equations in Year 8 — a *multifaceted variable approach*.

In earlier research (Tahir, Cavanagh, & Mitchelmore, 2009), found that students who were taught using a multifaceted variable approach did attain a deeper understanding of the variable concept. In this paper, we investigate whether this improved understanding of variables helped the students in solving linear equations. The hypothesis is that students who are taught using a multifaceted approach will be more successful in solving linear equations. To test this hypothesis, the equation solving strategies of students taught using the multifaceted variable approach were compared with those used by students who were taught in a traditional way.

Solution Strategies

When students are presented with an equation such as $x + 5 = 8$, they usually see it as an arithmetic process and they prefer to use guess and check or working backwards as a strategy to solve this equation. It is not until they come across an equation of the type $2x + 5 = x - 7$, with x on both sides, that they are forced to think of an equation as an object to be operated upon to solve it (Sfard & Linchevski, 1994). Kieran (1992) presented a summary of strategies used by students to solve a linear equation, namely, known facts, counting techniques, guess and check, cover up, working backwards, and formal operations. These equation solving strategies can be arranged from least to most sophisticated as guess and check, using known facts/counting strategies, inverse operations, working backwards then guess and check, working backwards then known fact, working backwards and transformations (Linsell, 2009). The most sophisticated strategy of transformations is understood by very few students (Linsell, 2009).

Method

The study was conducted in a girls’ secondary school in Sydney where some teachers used our multifaceted variable approach to teach algebra. This two year, longitudinal study was completed in two phases. Phase 1 was conducted with students and teachers of Year 7 and focused on the concept of variable acquired by the students. The error analysis indicated that experimental classes demonstrated a deeper conceptual understanding of variable as compared to the comparison classes in Year 7 (Tahir, Cavanagh, & Mitchelmore, 2009). Phase 2 was conducted with the same cohort of students then promoted to Year 8, and with their same teachers when the classes were learning how to solve simple linear equations. Phase 2 focused particularly on the students’ solution strategies since they are not only alternative approaches to solving equations but they also represent different stages of conceptual development (Filloy & Sutherland, 1996).

Sample

The sample consisted of four classes graded by the school on the basis of their mathematical ability at the beginning of Year 7. Students of Set1 (high ability, 26 students) and Set3 (medium ability, 26 students) formed the comparison group and students of Set2 (medium Ability, 27 students) and Set4 (low ability, 19 students) formed the experimental group. The experimental group was taught using the multifaceted variable approach; they studied three aspects of variable (unknown, generalised number, and functions) in Year 7 before moving on to manipulation and solution of linear equations using real contexts in

Year 8. The comparison group studied patterns for generalisation but spent most of their Year 7 lessons learning to manipulate and simplify algebraic expressions. In Year 8 they also studied symbol manipulation and the method of solving simple linear equations.

Phase 2

Phase 2 covered the Year 8 algebra lessons. A meeting with the teachers of experimental group took place at the beginning of this phase. Results of Phase 1 were discussed and "Working mathematically: Patterns and Algebra" Workbook B (McMaster & Mitchelmore, 2008) was given to the teachers of experimental group. The syllabus arrangement of both groups is given in Table 1.

Table 1
Syllabus arrangement of classes

Sample	Topics
Comparison Group	Translate between words and algebraic symbols, expand, simplify and factorise algebraic expressions and fractions including expressions with indices, solution of simple linear equations
Experimental Group	Algebra in spreadsheets, simplify and factorise algebraic expressions and fractions, solution of simple linear equations

Set1 completed the entire Year 8 algebra course in seven weeks (six lessons/week) whereas Set2, Set3 and Set4 required an additional two weeks (eight more lessons) to complete the algebra course. During these two weeks Set1 was involved in enrichment work in algebra, including algebraic fractions, linear inequalities and solving simultaneous equations.

After nine weeks of algebra teaching, a test designed by their teachers in consultation with the first author was administered to all students. Set1, Set2 and Set3 was given the same test but some questions were replaced by easier questions for Set4 (the low ability class). This test included multiple choice, short response and extended response questions on algebra and geometry. In the algebra section, students were given two multipart, short response questions targeting algebraic manipulations (addition, subtraction, multiplication, division, factorisation, and expansion) and two separate questions where they were required to solve simple linear equations. One week after the algebra test, students were given a separate 15 minute algebra quiz designed by the first author. This quiz was intended to assess the skills of identifying equivalent equations as well as transforming one equation into another to show the equivalence. With easy access to computer algebra systems (CAS) and calculators which can solve a simple linear equation, the recognition of equivalent equations and the skill of transformation of an equation into another has become central for success in algebra (Ball, 2001).

During algebra teaching, one lesson per week of each class was observed by the first author. After the algebra test, 5-6 students of varying ability, selected by their teachers, were also interviewed by the first author. This paper will focus on similarities and differences between the experimental and comparison group in the selection of solution strategies for solving a linear equation, in order to investigate which approach might be more suitable for teaching linear equations.

Results and Discussion

The linear equations solved by students in the algebra test given after nine weeks of algebra teaching test are listed in Table 2 as Q1. Table 2 also shows as Q2 the quiz given a week after the algebra test.

Table 2
Linear equations solved by all classes in algebra test.

Q1 Solve

i) $n + 6 = 4$
ii) $7x = 56$
iii) $\frac{p}{5} = 9$
iv) $3(m - 1) = 18$
v) $\frac{p}{5} + 6 = 9$
vi) $7x - 2 = 5x + 8$

Q2 Which of the following equations can be transformed to $x - 2 = 0$? For the ones that can be transformed to $x - 2 = 0$, show how you realised this.

a) $2x = 4$
b) $4 = 2x$
c) $\frac{x}{2} = 4$
d) $4x = 2$
e) $x + 1 = 3$
f) $x - 3 = 1$

For Q1, student responses were allocated a score of 1 for correct and 0 for incorrect, and a total score was calculated by adding their marks in all six equations. One way ANOVA indicated significant differences between all four participating classes ($F(3, 86) = 6.10$, $p<0.001$). A Bonferroni post hoc test showed no significant difference between Set1 and Set2 or between Set3 and Set4, all other differences being significant.

The mean of all participating students considered as one sample indicated that the linear equations in Q1 were numbered in order of the difficulty level. Moreover, the simple one step linear equations given in part i, ii and iii of Q1 could be classified together at a lower difficulty level and the two-step linear equations given in part iv, v and iv were at a comparatively higher difficulty level (see Table 3).

Table 3
Mean and Standard Deviation of all students

Equation	n+6=4	7x=56	p/5=9	3(m-1)=18	p/5+6=9	7x-2=5x+8
Mean	0.97	0.94	0.93	0.69	0.65	0.63
Standard Deviation	0.17	0.24	0.26	0.47	0.48	0.49

Solution strategies used for Q1

The solution strategies selected by students were the balancing method, working backwards, guess and check, and some students just gave their answer without any calculation or justification.

1. Balancing method (B): for example $2x = 4$, dividing both sides by 2 results in $\frac{2x}{2} = \frac{4}{2}$ giving answer as $x = 2$.

2. Working Backwards/Inverse Operations (WB): for example $2x = 4$ (inverse of multiplications is division therefore) $x = \frac{4}{2} = 2$ or for $x + 1 = 3$, $3 - 1 = 2$ therefore, $x = 2$.
3. Guess and Check (G&C)/Known fact: for example: $2x = 4$ as $2 \times 2 = 4$, $x = 2$ or $x - 3 = 1$ as $4 - 3 = 1$ therefore $x = 4$.
4. Answer without justification (Ans): answer given without any calculation or justification.

The percentage of all student responses in each category, pooled across all parts of Q1, was calculated for each participating class. The resulting percentages are represented in Figure 1. Recall that Set1 and Set3 were the comparison classes and Set2 and Set4 the experimental classes.

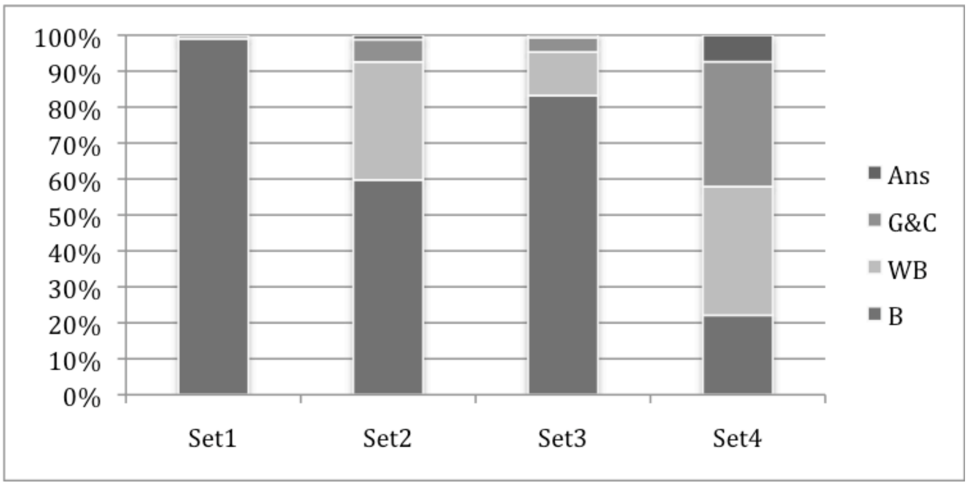

Figure 1. Distribution of various solution methods used by participating classes in Q1 (B: Balancing method, WB: Working backwards, G&C: Guess and check, Ans: Answer).

In Set1, all students used the balancing method to solve all of the linear equations, except for one student who chose working backwards in part iv of Q1. It is worth noting that the balancing method was the only method which was used by their teacher in algebra lessons. In Set3, the main method selected to solve linear equations was again the balancing method (74% of responses). Some students who were not successful in using the balancing method reverted back to arithmetic methods such as guess and check (12%) and working backwards (4%) This class was also taught using the balancing method to solve equations in their lessons and guess, with check as an alternate way of finding the solution of a simple linear equation. In some instances the teacher also explained the balancing method by demonstrating the procedure of working backwards.

The teachers of Set2 and Set4 used the given teaching resource (McMaster & Mitchelmore, 2008) which focused on the meaning of variable in each problem. Students in experimental group (Set2, Set4) mostly used the method of inverse operations and less time was spent on the balancing method.

Despite these differences, the comparison groups (success rate: 85%) and experimental groups (86%) were equally successful in using the balancing method to solving the given equations. For example, 89% students of Set2 were able to correctly solve part vi of Q1, as compared to 81% students of Set1. It appeared that the experimental group's extensive use of the working backwards strategy had helped them understand and use the balancing method as accurately as the high ability class.

It is interesting to note here that the percentage of responses in Set2 that showed the balancing method was smaller than the percentage of such responses in Set3. Therefore it is not always necessary that high ability students automatically choose a more sophisticated method to solve an equation. Furthermore, the selection of solution strategy made by the experimental classes depended upon the equation to be solved. Sometimes the students in the experimental classes were more successful in selecting a suitable strategy to correctly solve linear equations than those in the comparison group; for example, Set 4 used guess and check in part v of Q1 more successfully than Set3, who mostly used the balancing method).

Students used the following three methods to solve Q2, namely the transformation method, the substitution method, the comparison method. Also, some students just solved the equations one by one and did not select any equation as an answer.

1. Transformation method (T): Transform each equation one by one (using balancing method or inverse operations/working backwards). For example: $x + 1 = 3$, subtract 3 from both sides to get $x + 1 - 3 = 3 - 3$, thus $x - 2 = 0$. Or solve the equation first as $x + 1 = 3$, $x + 1 - 1 = 3 - 1$, $x = 2$, and then transform as $x - 2 = 2 - 2$ giving $x - 2 = 0$
2. Substitution method (S): Solve equation $x - 2 = 0$ as $x = 2$, then substitute $x = 2$ in the given equations one by one; if the equation is true, then mark it as the answer. Or solve each equation and substitute answer in equation $x - 2 = 0$ to verify.
3. Comparison method (C): Solve the equation $x - 2 = 0$ as $x = 2$ first, then solve other equations one by one using any method such as balancing method, working backwards or substitution, .if the answer is also $x = 2$ that equation is marked as the answer.
4. Solve the equations by any method of choice like balancing, working backwards, guess and check or substitution then a) select some equations as equivalent to the given equation x-2 = 0 without giving any justification (Answer without justification (Aj)) or b) no equation selected as an equivalent equation (Equation not identified (Ai)).
5. Equivalent equations identified without solving the equation (As): Equivalent equation selected without solving the equation.
6. Not solved or incorrectly solved (NA): Equation not solved or incorrectly solved and no equation is selected as answer.

The percentage of student responses in each category, pooled across the various parts of Q2, was calculated for each class and is shown in Table 4.

Table 4
Percentage of student responses using various methods in Q2

	T	S	C	Aj	Ai	As	NA
Set1	13	63	5	16	0	1	1
Set2	16	29	6	33	11	3	2
Set3	7	19	1	56	0	15	1
Set4	8	2	0	17	55	0	18

Note: T: Transformation method, S: Substitution method, C: Comparison method, Aj: Answer without justification, Ai: Equation not identified, As: Answer without solving equation, NA: Not solved or incorrectly solved and no equation selected as answer.

Students who used the transformation method were operating on an equation algebraically rather than numerically, and slightly more responses in the experimental group (22%) crossed this boundary than in the comparison group (20%). The percentage of responses which used the substitution method decreased with the ability level of the class. Note that the comparison classes started algebraic manipulations with finding unknowns in an algebraic expression in Year 7 and then they moved on to the balancing method to solve a linear equation. Thus the inclination to solve each equation mainly by using the balancing method and then using a substitution method to answer the Q2 was most likely due to the frequency of these two methods in their lessons.

The mean percentage of students selecting an equation as their answer without giving justification was highest in Set3 (56%) and not giving any answer after solving the equation was highest in Set4 (55%) (not surprising, as this class had not studied the topic of equivalent equations in their lessons). Set1 and Set3 have studied equivalent equations before the quiz and Set2 had also completed this topic during their algebra lessons.

Discussion and Implications

Students' selection of a solution strategy depended extensively on the strategies employed by their teachers to solve linear equations in their lessons. The experimental group selected a solution strategy which they thought was more suitable to solve the given equation, and they were more successful in solving some linear equations as compared to the comparison classes. For example, 74% students in Set2 chose the balancing method to solve $7x - 2 = 5x + 8$ and 89% of them were able to solve this question correctly, as compared to 81% students of Set1 — despite the fact that Set1 was the highest ability class who also studied advanced topics such as simplifying expressions with indices, factorisation of algebraic fractions and solving simultaneous linear equations. This result suggested that the experimental group was not automatically solving each question by the same method practiced in the class. They were dealing with each equation on its merit.

The comparison classes used the balancing method throughout their lessons. However, more students in the comparison classes used substitution or comparison methods to show the equivalence of two equations. This finding suggests that they were associating equivalence of equations with finding a common numerical solution of the given equations. Whereas the experimental classes had spent less time on manipulation of algebraic terms and on solving equations by the balancing method, still more students in the experimental group used a transformation method to show the equivalence of two equations as compared to the comparison group. It was also very encouraging that some students in the low ability experimental class linked equivalence of an equation with finding a transformed equation.

The results of Phase 1 showed that students who used a multifaceted variable approach acquired a deeper understanding of variable (Tahir, Cavanagh, & Mitchelmore, 2009). The results of Phase 2 suggest that they also formed a deeper understanding of equations, and were equally successful in solving linear equations as the comparison group. The experimental group came in contact with problems based on real contexts, and had to think about the variable involved and decide on a suitable solution strategy. The reinforcement of a strategy by solving many exercises may give students an advantage in getting good marks, however this does not mean that they are learning algebra with understanding. It appears that a multifaceted variable approach does help students learn algebra with understanding.

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