Upper Primary School Students' Algebraic Thinking

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This qualitative research study involving 128 students in grades 4-6 was conducted to develop a framework for characterizing upper primary school students' algebraic thinking. Four levels of algebraic thinking were identified from student responses to tasks involving patterns and open number sentences. Level 1 students failed to understand the tasks or answered with irrelevant data. Those at Level 2 understood the tasks but were unable to proceed further. Level 3 students were able to complete the tasks but were unable to link one aspect of the task to another. Level 4 students understood the relationship among various aspects of data and used all aspects of the data.

To improve students' learning in mathematics, it is necessary to understand the developmental mode of their thinking and reasoning. With the nature of mathematics that deals with abstract entities, students have difficulty in understanding mathematics concepts, especially those in algebra. Therefore, thinking, particularly algebraic thinking, is a tool for understanding abstraction (Russell, 1999). Typically, most school mathematics curricula separate the study of arithmetic and algebra. Arithmetic is the primary focus of elementary school mathematics and algebra is the primary focus of secondary school. So, it is difficult for students to transform from arithmetic to algebra. Students in later grades of elementary school should have more experiences of formal study of algebra (Kieran, 2004; National Council of Teachers of Mathematics [NCTM], 2000). A broader conception of algebra emphasises the development of algebraic thinking, rather than just the skilled use of algebraic procedures. Although algebra is not the focus of the *primary* school curriculum, students should be prepared to be familiar with algebra especially generalization. Although algebraic thinking, a tool for learning algebra, is one of the themes for developing students' understanding of mathematics, there are few suggestions of what should be used in the classroom activities, especially in Thailand. The researchers were interested in two key components of the elementary school mathematics, namely, patterns and open number sentences. Existing research on primary school students' algebraic thinking has not yet generated a framework for systematically characterizing students' algebraic thinking based on the cognitive learning theory such as the SOLO (Structure of the Observing Learning Outcome) model of Biggs and Collis (1982). In essence, if an algebraic thinking framework is developed, it would provide detailed cognitive models of students' learning that can guide the construction and planning of mathematics instruction and curriculum as Cobb (2000) have suggested.

Purpose of the Study

The purpose of this study was to formulate a framework to characterize upper primary school students' algebraic thinking in the content areas of patterns and open number sentences.

Theoretical Consideration

Biggs and Collis developed the SOLO Model which suggests five modes of functioning*; sensorimotor, ikonic, concrete-symbolic, formal, and postformal* functions.

L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia.* Fremantle: MERGA.

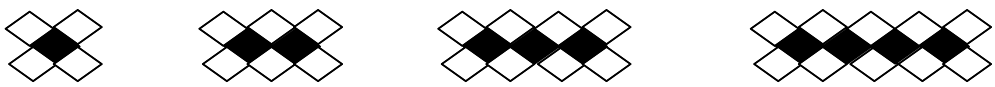
Each mode consists of five levels of thinking; *prestructural, unistructural, multistructural*, *relational* and *extended abstract* levels. The SOLO Model and previous studies related to mathematical thinking or frameworks for characterizing children's mathematical thinking (Biggs and Collis, 1982; Jones, Thornton, Langrall, Mooney, Perry, & Putt, 2000; Mooney, 2002) were consulted in order to formulate a framework for characterizing upper primary school students' algebraic thinking. Since this study focused on upper primary school students, the researcher hypothesized that students were on the concrete symbolic mode of SOLO model. This concrete symbolic mode involves a more abstract process of learning and is considered as a significant shift in abstraction, from direct symbol systems of the world through oral language to the use of second order symbol systems such as written language that can be applied to the real world. Within this mode, even though the SOLO model characterized students into five levels, several previous studies (Jones, Langrall, Thornton, & Mogill, 1997; Jones, et al., 2000; Mooney, 2002) indicated that students responded only to four levels of thinking. They did not respond beyond relational level. Thus, in this study the researcher expected that students can be characterized in four levels of algebraic thinking: *prestructura*l (Level 1), *unistructural* (Level 2), *multistructural* (Level 3), and *relational* (Level 4). The upper primary school students' algebraic thinking is referred to the ability of students to use their thinking skills to generalize the patterns and analyze a relationship between numbers on each side of the equal sign.

Methodology

Data Collection and Validation

After an initial framework was developed, the researcher conducted a test and an interview guide in order to collect data to validate the initial framework. The test was tried out with six students as a pilot group, two from each grade, to determine the appropriateness (language and difficulty) of the test and to study the ways students responded to questions. After that, the test was revised again for the final form, illustrated in Figure 1.

<u>The Pattern task</u>



From the bead pictures, answer these questions and show the way of thinking that you use to answer the question.

(1.1) Making beads to be as the pictures by using 5 black beads.
How many white beads are used?

(1.2) Making beads to be as the pictures by using 30 black beads.
How many white beads are used?

(1.3) Making beads to be as the pictures by using 100 black beads.
How many white beads are used?

<u>The open number sentence tasks</u>

1) 8 + 5 = ____ + 8
2) 32 + 45 = ____ + 30

Figure 1. Examples of pattern task and open number sentence tasks.

The interview guide was trialled with the pilot group. The results of the pilot study were transcribed and coded by three coders (the researcher and two coders). The coders were trained by the researcher before they did the coding. The test consisted of three pattern tasks and an open number sentence task incorporating four questions (see the example of a pattern task and the open number sentence tasks in Figure 1). The test was administered to 128 upper primary (4^{th} to 6^{th}) grade school students from three classrooms. Eighteen of the 128 students were chosen for interviewing in order to refine the descriptors in an initial framework. Six students were chosen from each of the three grades. Two students from each grade were selected from the high, average and low achievement groups.

In the process of validating the framework, each of these 18 students was interviewed by the researcher to gain further insight into their thinking. After that, the interviews were transcribed and coded with the students' paper work. Three coders independently coded all 18 student interviews and responses from the paper work. In coding data, the coders based their work on a double coding procedure suggested by Miles and Huberman (1994) and then verified their differences until a consensus was reached. The reliability among the coders was 84%. The results from analysis were used to refine and validate the algebraic thinking framework.

Results

The results of this study consist of two phases. The first phase presents the formulation of the algebraic thinking framework and the second phase presents the results of validating the algebraic thinking framework. This paper describes only the result from the second phase.

Refinements to the Initial Framework

The researcher refined the framework to improve consistencies and eliminate discrepancies between the descriptors in the framework and students' responses. Table 1 shows the descriptors of the refined algebraic thinking framework across two key indicators. Underlined statements are those modified in the initial framework so as to represent more closely students' responses in algebraic thinking.

Profiles of Students' Levels of Algebraic Thinking

Based on the results of coded responses, a student was assigned to a level of thinking for each key algebraic situation following agreement of coders and the researcher. Figure 2 shows profiles of 18 students' levels of algebraic thinking for two key components.

From these profiles, 13 students (72.22%) had the same levels of thinking across all two key components. Five students (27.78%) had levels of thinking with one component one level higher or one level lower only.

Table 1

Refined upper primary school students' algebraic thinking framework

Indicator / Level	**Pattern**	**Open Number Sentence**
Level 1	**P11**: Unable to explain how the terms in the pattern relate. **P12**: Unable to find the next term, the near term, the far term, and general term of the given pattern. **P13**: <u>The reason used to find the answer for the pattern is generated through guessing or citing irrelevant evidences.</u> **P14:** The explanation of the reason is based on empirical method – choosing only a part of the given data to form the conclusion, e.g. using the data from a particular term of the pattern to answer.	**S11:** Look at the equation in new perspective by transforming it in to "Problem = ___ " **S12**: Not concerned if both sides of the equal sign are equal disabling them to analyse that the number put in the blank must be the one that makes both sides of the equal sign equal. **S13**: Unable to analyse the function of each of the numbers in the equation nor see the function of each number in the open equation. **S14**: Find the answer by taking all the given numbers in the open equation to operate and put the product in the blank without concerning in what position the blank is. Such operation is done under the transformation of the open equation into the new one. That is " – Problem = ____"
Level 2	**P21**: Able to analyse one-dimensional function of the terms in the pattern. **P22:** Unable to analyse two-dimensional function of the positions of the term and value of the term in such positions. **P23:** Able to find the next term, and the near term of the given pattern but unable to find the value of the far term and general term of the given pattern. **P24:** The acquisition of the conclusion is done through conceiving the recursive function as the value of the next term = value of the preceding term + difference between the terms. That is the preceding term is used to find the value of the following term.	**S21:** View the given open equation in term of "Problem = answer" without being concerned with other numbers in the same side with the answer (if any) **S22:** Not concerned if both side of the equation is equal disabling them to analyse the numbers to be put in the blank which would make the both sides equal. **S23:** Unable to analyse the relationship between number nor see the relationship between each number in the open equation. **S24**<u>: Find the answer by taking the number in the left side of the equation to add together or to subtract from one another </u>.
Level 3	**P31**: Unable to analyse the function of the value of each term in the pattern of the value of the following term = the value of the preceding term + the difference between the terms. **P32:** Able to analyse the function between the positions of the term and value of the term in such positions by using induction – to find the formula to represent function that is relevant to the value of the given preceding terms and conclude that the proposed formula is valid for each term regardless of the origin or definition of the numbers used to validate the formula and able to use the formula to find the value of the terms in particular conditions.	**S31:** View the open equation as "product on the left hand = product on the right hand" **S32:** Able to analyse that the number to put in the blank of the equation must be the one that makes both side of the equal sign equal. **S33:** <u>View the open equation in parts disabling them to analyse the relationship between numbers in the open equation nor see the relationship between numbers of the open equation.</u> **S34:** Find the answer by using computational reason. That is to compute the product on one side of the equal sign and subtract it from the remaining numbers on the other side of the equal sign.

Indicator / Level	**Pattern**	**Open Number Sentence**
Level 3	**P33:** Able to find the formula to represent the general terms of the given pattern enabling them to find the value of the following term, the near term, and the far term, and any other terms of the pattern.	
Level 4	**P41:** Able to analyse the function of the terms in the pattern of value of the following terms = value of the preceding term + difference between the terms. **P42:** Able to analyse the function of the position of the terms and value of the term at each position by proving through the condition of the given pattern in a meaningful and logical ways not through the induction. Students at this level could clearly explain the origin and meaning of the numbers appeared in the general formula of the pattern. **P43**: Able to find the formula to represent general terms of the given pattern enabling them to completely find the following term, the near term, the far term, or any other terms in the pattern.	**S41:** View the given open equation in terms of the left side expression = the right side expression. **S42:** Able to analyse that the number to be put in the blank of the open equation must be the one that makes both side of the equal sign equal. **S43:** View the open equation as a whole, not separately look at each part, enabling them to see the relationship between numbers in the open equation. **S44**: Find the answer by using functional logic through the analysis of the relationship of the numbers in the open equation.

Analysis of Algebraic Thinking at Each Level

In order to generate a final picture of each of the levels of students' algebraic thinking, the researchers chose a student to represent those who had the same level of thinking across two key algebraic situations to illustrate the point.

Level 1 Thinking. Students exhibiting Level 1 algebraic thinking were confused or unable to understand the tasks. They avoided answering the question or answered by guessing basing on irrelevant data.

Wasu, who was studying in 4th Grade, was selected to illustrate a Level 1 algebraic thinker. For the *pattern* problems, Wasu did not understand or was confused about the pattern. He gave an irrelevant answer. He did not know that the given data was a pattern. He did not know how to use the given pattern to answer the question. When finding a specific term as the answer, he could not find any term of a given pattern. In Problem 1 of pattern problems, when asked to find the number of black beads with five white beads, Wasu drew a new figure irrelevant to the given one to answer the question. For the open number sentence, Wasu could read the questions but was unable to understand the questions, especially those with an equals sign. He showed misconceptions about the equals sign, explaining that it was as an operator meaning "the total". When he found a missing number of any open sentence, he carried out the operation with all given numbers such as 8 + 5 = ___ + 8 or 3 + 6 + 9 = 3 + ___ . He answered by counting all given numbers on the question. With question (1) 8 + 5 = ___ + 8, he gave 21 as the answer because he added 8 with 5 and 8. To answer question (2) 3 + 6 + 9 = 3 + ___ he carried out the operation with 3, 6, 9 and 3. He was not concerned with the equals sign.

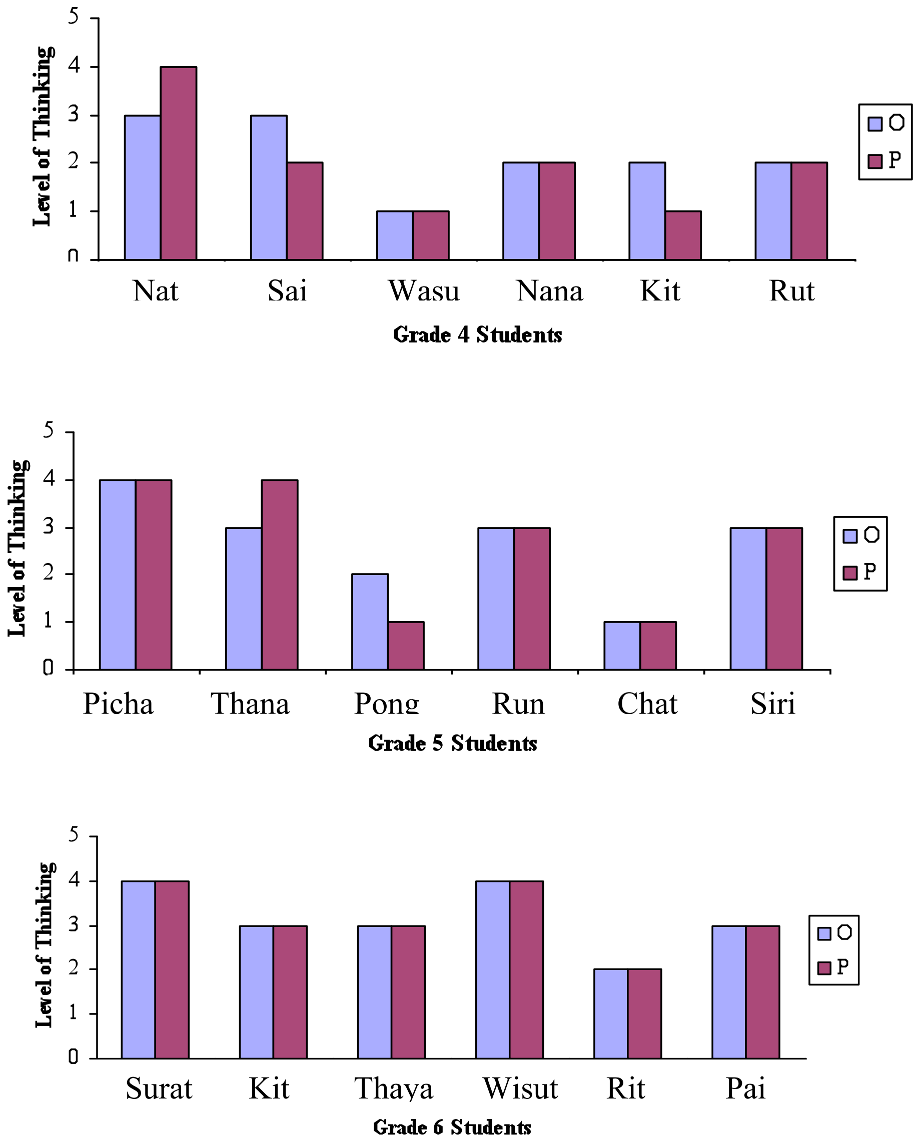


*Figure 2***.** Profiles of Students' Levels of Algebraic Thinking.
(O = Open number sentence; P = Pattern)

Level 2 Thinking. Students with level 2 thinking demonstrated thinking beyond Level 1. They could engage with the tasks and understood the requirement of the tasks~~,~~ but were not able to proceed further.

Rut, who was studying in 4th Grade, was selected to represent those with Level 1 algebraic thinking. With the *pattern* problems, Rut could find the next term by using a recursive idea. Nevertheless, he could not find the greater or general term of a given pattern. For the first pattern problem, Rut could find the number of black beads with five white beads by drawing the beads like the given pattern and then count the number of the black beads. He recognized that there was a growing pattern with two black beads. When asked to find the number of black beads with 30 white beads or with 100 white beads, a higher term, Rut used the same method to get the answer. This is a wrong answer because his picture was not clear so it was difficult to count the number of black beads. For the *open number sentence*, Rut confused the equals sign with an operator meaning "the answer”. He found a missing number of an open sentence by computing the given number on the left side of equals sign with the reasonable conclusion that the right side of an open number sentence had only one number. He was not concerned with the given number on the right side of the equals sign.

*Level 3 Thinking***.** Students with level 3 thinking demonstrated an ability to complete the tasks but were not able to link one aspect of the task to another.

Siri who was studying in 5th Grade was chosen to represent Level 3 algebraic thinkers. With the pattern problem, she could find the general relation between terms and position of a term in a giving pattern by using inductive reasoning. With an open number sentence, Siri showed correct understanding of the equals sign. She regarded the equals sign as a relationship between the numbers on each side of the equals sign. She could find the missing number of any open number sentences by using computation.

*Level 4 Thinking***.** Level 4 students were able to see relationships between the given data and demonstrated a meta-cognitive understanding of the relationship among various aspects of data. They also used all aspects of data when solving the problems.

Wisut, who was studying in 6th Grade, was chosen to represent students who were at Level 4 across two components. With the pattern problem, he could prove the general relation between terms and position of a term in a given pattern. He could clearly interpret the meaning of the terms he used. With the open number sentence, he showed his thinking about the equals sign on open number sentence as the relational sign and used the relationship between the numbers on each side of the equal sign to find the missing number of any open sentences correctly which is called ‘relational’ thinking.

Discussion

The profile of students’ levels of algebraic thinking showed strong consistency across the two components. 72.22% of the sampled students demonstrated consistent thinking levels across the two components. The consistency of this framework compared favourably with the evidence in a number of other studies on cognitive frameworks (Jones, et al., 1997; Tarr & Jones, 1997; Mooney, 2002). This consistency confirms that the framework provided a cohesive picture of upper primary school students’ algebraic thinking.

In this study, a framework for characterizing lower secondary school students’ algebraic thinking was formulated and validated. It could be claimed that the validated framework reflected a coherent picture of students’ algebraic thinking offering insight into how algebraic learning of upper primary school students develops. While no claim has been made that the framework is applicable to all students, it provides the teachers with knowledge of students’ algebraic thinking that is applicable well beyond the classroom in which the study was conducted. As a result, this landscape of upper primary school students’ algebraic thinking can be used by curriculum developers and teachers to understand situations of instruction and assessment (Cobb, 2000; Fennema & Franke, 1992).

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References

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the Quality of Learning: The SOLO Taxonomy (Structure of Observed Learning Outcomes)*. New York: Academic Press.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In Anthony E. Kelly and Richard A. Lesh (eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 307-334). Mahwah, NJ: Lawrence Erlbaum Associates.

Fennema, E., & Franke, M. (1992). Teacher's knowledge and its impact. In Douglas A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning*, (pp. 147-164). New York: Macmillan.

Jones, G.A., Langrall, C.W., Thornton, C.A., & Mogill, A.T. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics*, *32*, 101-125.

Jones, G.A., Thornton, C.A., Langrall, C.W., Mooney, E.S., Perry, B. & Putt, I.J. (2000). A framework for characterizing children's statistical thinking. in *Mathematical Thinking and Learning*, *2*(4), 269-307.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator* (Singapore), *8*(1), p. 139-151.

Miles, M., & Huberman, A., (1994). *Qualitative Data Analysis*. Thousand Oaks, CA: Sage.

Mooney, E. (2002). A framework for characterizing middle school students' statistical thinking. *Mathematical Thinking and Learning. 4*(1), 23-63.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Russell, S. (1999). Mathematical Reasoning in the Elementary Grades. In L. V. Stiff and F. R. Curcio (eds.), *Developing Mathematical Reasoning in Grades K-12*, (pp.1-12), Reston, VA: National Council of Teachers of Mathematics.

Tarr, J., & Jones, G. (1997). A framework for assessing middle school students' thinking in conditional probability and independence. *Mathematics Education Research Journal*, *9*, 39-59.