‘You might say you’re 9 years old but you’re actually *B* years old because you’re always getting older’: Facilitating Young Students’ Understanding of Variables

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Student transition from arithmetical understandings to algebraic reasoning is recognised as an important but complex process. An essential element of the transition is the development of a rich understanding of variables. Drawing on findings from a classroom-based study, this paper outlines the instructional tasks and pedagogical actions a teacher used to facilitate her students understanding of variables. The findings affirm that younger students can begin developing understanding of variables and use forms of algebraic notation to represent their mathematical ideas. Carefully designed tasks, specific pedagogical actions and extended discourse were all important elements in facilitating student understanding.

Over the past decade, significant changes have been proposed for mathematics classrooms in order to meet the needs of a knowledge society. One aspect of this change has been an increased focus in both research and curricula reform on the teaching and learning of early algebraic reasoning. This emphasis in part is due to the growing acknowledgment of the insufficient algebraic understandings of many students and the way in which this denies them access to potential educational and employment prospects. To address the problem, one response has been to integrate teaching and learning of arithmetic and algebra as a unified curriculum strand in policy documents (e.g., Ministry of Education 2007, National Council of Teachers of Mathematics 2000). Within this strand, the combination of students’ informal knowledge and numerical reasoning can be used to transition early algebraic thinking. Essential to this transition is a requirement that students develop rich understanding of variables. However, developing such understanding in primary classrooms is challenging. Therefore, the research reported in this paper examines the instructional tasks and pedagogical actions a teacher used with a class of 9 to 11 years olds in order to facilitate them to represent their mathematical ideas using notation and gain a more sophisticated understanding of variables.

In the transition from arithmetical to algebraic reasoning, a deep understanding of variables is important. We know, however, that many young students have limited classroom experiences in exploring variables (MacGregor & Stacey, 1997; Weinburg et al., 2004). A large-scale exploratory study of Year 7-10 students by MacGregor and Stacey suggested that student difficulties could be attributed to both the lack of opportunities to explore variables and their classroom experiences. These researchers found that inappropriate teaching methods such as the use of letters to represent an object led to students viewing letters as abbreviated words. Also a number of the students in the study based their interpretation of symbolic letters on intuition, guessing or false analogies. Similarly, a study by Knuth and his colleagues (2005) with 6th to 8th graders highlighted a range of common misconceptions linked to notation. These included the notion that a single letter variable could only stand for a single number and variables represented by different letters could not be the same number.

We know that students’ algebraic understandings develop with experience (Flockton, Crooks, Smith, & Smith, 2006). Providing opportunities in the classroom for students to

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both use notation and explore the concept of variables supports them to deepen their understanding. A study by Weinburg and his colleagues involving middle school students demonstrated that performance in interpreting algebraic notation and understanding of variables as representing multiple values increased over year levels. Other recent research with primary age students advocates that younger students are able to understand and work with algebraic notation. Results of an interview based study carried out with third grade students by Schlieman and her colleagues (2007) found that the students could develop consistent notations such as circles or shapes to "represent elements and relationships in problems involving known and unknown quantities" (p. 59).

Studies involving classroom interventions provide examples of how young children can use variables as a tool to understand and express arithmetical and functional relationships. A classroom intervention study by Carraher and his colleagues (2006) involving third grade students found that the students were able to use formulas to represent functions and treat the symbolic letter in the additive situation as having multiple possible solutions. Another study by Stephens (2005) with Year 7 and 8 students illustrated how a mathematical problem could be used to confront common misconceptions students held about variables. Teachers in the study noted that concentrating on symbolic representations allowed them to address misconceptions. Carpenter and his colleagues (2005) worked with students across the primary grades and found that students who had worked with number sentences were able to easily adapt these to represent generalisations. At the conclusion of the study, 80% of 4^{th} and 6^{th} Graders were able to use variables to express generalisations.

The theoretical framework of this study draws on the emergent perspective (Cobb, 1995). From this socio-constructivist learning perspective, Piagetian and Vygotskian notions of cognitive development connect the person, cultural, and social factors. Therefore, in this paper the learning of mathematics is considered as both an individual constructive process and also a social process involving the social negotiation of meaning.

Method

This research reports on episodes drawn from a larger study, which involved a 3-month classroom teaching experiment (Cobb, 2000). The larger study focused on building on numerical understandings to develop algebraic reasoning. It was conducted at a New Zealand urban primary school and involved 25 students aged 9-11 years. The students were from predominantly middle socio-economic home environments and represented a range of ethnic backgrounds. The teacher was an experienced teacher who was interested in strengthening her ability to develop early algebraic reasoning within her classroom.

At the beginning of the study, student data on their existing numerical understandings was used to develop a hypothetical learning trajectory. Instructional tasks were collaboratively designed and closely monitored on the trajectory. The trajectory was designed to develop and extend the students' numerical knowledge as a foundation for them developing early algebraic understandings. This paper reports on the tasks on a section of the trajectory that focused on developing understanding of variables. The students were individually pre and post interviewed using a range of tasks drawn from the work of other researchers (e.g., Knuth et al., 2005, Weinberg et al., 2004). The rationale for selecting these questions was to replicate and build on the previous findings of these researchers. Other forms of data collected included classroom artefacts, detailed field notes, and video recorded observations.

The findings of the classroom case study were developed through on-going and retrospective collaborative teacher-researcher data analysis. In the first instance, data analysis was used to examine the students' responses to the mathematical activity, and shape and modify the instructional sequence within the learning trajectory. At completion of the classroom observations the video records were wholly transcribed and through iterative viewing using a grounded approach, patterns, and themes were identified. The developing algebraic reasoning of individuals and small groups of students was analysed in direct relationship to their responses to the classroom mathematical activity. These included the use of tasks, the climate of inquiry, and the pedagogical actions of the teacher.

Results and Discussion

I begin by providing evidence of the initial understandings of the students. I then explain the starting point for the section of the trajectory related to variables. The initial starting point for classroom activity is outlined and I explain how this was used to develop student understanding. A sample of the varied activities, which the students engaged in to explore variables, is provided. I conclude with evidence of the effect of the classroom activities using post student interview data.

Interview Data of the Student's Initial Concepts of Variables

This section presents pre-task interview results. One interview item[15] investigated the use of notation to represent an unknown quantity. Students predominantly used a specific number to represent the unknown quantity (see Table 1). These results demonstrate students' unfamiliarity or reluctance to symbolically represent an unknown quantity.

Table 1
Percentage of Students (n =25) Using Notation for an Unknown Quantity

	Correct notation e.g., ■+ 3 $N + 3 + 2$	Non-standard or incorrect notation e.g., $B + 3 = A$, $A + A = C$	Number as notation e.g., 2 + 3 = 5	No response
Q. A	24%	4%	60%	12%
Q. B	28%		60%	12%
Q. C	16%	12%	60%	12%

Students working at a higher level [n = 19] were asked a further question[16] to elicit their understanding of what a letter meant in a mathematical context. In response to Question A, 6 students responded that the symbol could stand for four and correctly justified this response. However, seven students' constructed their interpretation of symbolic letters by guessing or false analogies. Most frequently, the letter was viewed as an abbreviated word:

Sangeeta: There is sort of a clue in that letter because the reason I said four is because four starts with an f.

[15] What is a mathematical statement or sentence to represent each of the following situations:

A) I have some pencils and then get three more.

B) I have some pencils, then I get three more and then I get two more.

C) I have some pencils then I get three more and then I double the number of pencils I have.

[16] $2f + 3$ A) Could the symbol stand for 4? B) Could the symbol stand for 37?

In response to Question B, three students stated that the symbol could stand for thirty-seven and justified this response. Ten students stated that the symbol could not stand for thirty-seven. Two students reported that a single letter could only stand for a single digit number.

Rachel: It would have to be ff. It has to be one number.

A further item[17] was used to probe their understanding. Only four students stated that the sentence was sometimes true and justified their response. Five students stated the number sentence could never be true and indicated that they considered different letters in an equation could not represent the same number.

Josie: M and P couldn't stand for the same thing.

The Stepping off Point on the Trajectory

An initial activity[18] involving the use of an algebrafied arithmetic problem was used as a context to engage students in dialogue about variables. The task structure of the word problem provided a context for students to give conceptual explanations of their notation while specific pedagogical actions were important tools to scaffold use of notation. In the first instance, the teacher supported the students to record in a systematic way to emphasise the patterns in the equations. For example, during a whole group discussion, she revoiced a student's explanation scaffolding her to record in a logical manner and placing an emphasis on the patterns in the equations:

Teacher: Nine plus something equals seventeen … what if the T-shirt was twenty dollars? What would the equation look like if was twenty dollars?

Specific questioning was used to focus student attention on identification of the constant and unknown in the equations.The teacher first directed the students to identify the similarities and then the differences. In response the students referred to the context of the problem providing a conceptual explanation:

Teacher: Talk to the person next to you about what has stayed the same in all of those equations?
Zhou: The nine because that's how much you have in the bank.
Teacher: What changes in each of those equations?
Gareth: The amount that you have to save.
Sabrina: The cost of the t-shirt.

In the final part of this activity students were asked to use algebraic notation to represent a generalised situation. Students used informal algebraic notation to represent the situation. During the following large group discussion two alternative representations were shared.

Rachel: [writes $\square + 9 = a$] We drew a box plus nine equals *a*.
Heath: [writes $z - 9 = x$] We did *z* take away 9 equals *x*.

[17] Is $h + m + n = h + p + n$ always, sometimes or never true?

[18] If you had $9 in your bank and wanted to buy a T-shirt for $17, how much do you need to save? What about if the T-shirt cost $20 or $26 or $40? Have a go at solving the problem … what changes and what stays the same? Can you find a way to write a number sentence algebraically that someone could use to work out how much they need to save no matter what the cost of the T-shirt?

In subsequent lessons, when solving a further algebrafied arithmetic problem[19], the students modelled their recording (see Figure 1) on the student explanation scaffolded by the teacher. This enabled them to quickly identify the unknowns and construct algebraic notation to represent the situation. For example, Heath constructed a solution strategy:

> Heath: [writes equation] Triangle divided by five equals spiral.

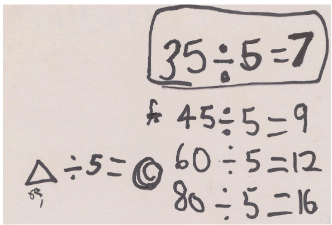

Figure 1. Solution strategy for the CD player problem.

Developing Further Understanding of Variables through Formalising Functional Rules

Further opportunities were provided for students to construct notations and extend their understanding of variables through the provision of functional relationship problems. For these tasks, students used variables to represent the rules and generalisations their groups had constructed for the functional relationships. For example, a group developed the following rule and explanation:

> Rachel: [writes A x 3 + 2 = Q] We did A times three plus two because you always times three and then you add two. We did A for the number of tables.

The teacher used this as an opportunity to further extend student use of formal notation introducing an algebraic convention:

> Teacher: [writes $3A + 2 = Q$] I can write it like this three A plus two equals Q because that's like putting brackets around here and plussing two because three A is the same as saying three times A.

The formalisation of generalisations into algebraic rules provided opportunities for students to develop their understanding of notation. For example, during small group work a student recorded a generalisation as ▲ x 5 = ▲. Another group member disagreed:

> Gareth: [pointing to both triangles] But these aren't the same numbers you need to change it from the triangle.

In another lesson, the activity[20] provided students the opportunity to extend their understanding of algebraic notation as a quantitative referent. During small group work, the students in one group discussed how their notation linked to the contextual basis of the functional relationship. Building on their notation S x 10 + 5 to represent the functional relationship, their discussion moved their understanding towards a generalised rule:

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19 You would like to buy a CD player that costs \$35. You earn \$5 an hour at your job. How many hours do you need to work? What about if the CD player costs \$45 or \$60 or \$80? Have a go at solving the problem … what changes and what stays the same? Can you find a way to write a number sentence algebraically that someone could use to work out how many hours they need to work no matter what the cost of the CD player?

20 Vodafone is currently offering a calling plan that charges 5 cents per call and 10 cents per minute..

1) How much would a 3 minute phone call cost? 6 minutes? 15 minutes?
2) Write a number sentence to show how much a phone call will cost no matter how long you talk for.

Tim: So S is meaning three.
Ruby: No it is not meaning only three. It is meaning the number of minutes you have had on the phone.
Tim: So at the moment it is meaning three minutes?
Ruby: No it is just meaning any number of minutes.

During the whole group discussion which followed the small group work, students also extended their understanding of formal notation. One student notated the generalisation as P x $10 + 5 = N$. Another student suggested using the teacher introduced shortened notation:

Rani: Instead of going P times ten couldn't you just go $10P$?

Several other students initially disagreed with this:

Bridget: No because it is a number and a letter.
Susan: You need to know if you are timesing it or plussing it.

Sensing an opportunity, the teacher again facilitated a discussion of using this notation:

Ella: What does it mean if you see a letter with a number in front of it? What does it mean in mathematics?
Steve: It's like ten times P because it is an algebraic short-cut.

Confronting Misconceptions About Variables

An important element of extending student understanding of variables was awareness of possible misconceptions. Midway through the study, an examination of an algebraic number sentence revealed that many students maintained the misconception that two different letters in an equation could not represent the same number. For example, during a whole class discussion a student argued that $J + T = T + L$ could never be true because:

Josie: L and J can't equal the same number … two letters can't represent the same number in the same equation.
Teacher: So you are saying that J and L can't represent the same number?
Josie: Yeah but T and T have to.

Other students agreed with Josie's argument.

Sabrina: I think they can if they are in different equations.
Josie: They can if they are in completely different equations … these are two equations which are joined so that means that they can't represent the same number.

Collaborative discussion between the teacher and researcher led to revision of the trajectory and the insertion of specifically designed algebraic number sentences[21] to confront this misconception. The first number sentence reinforced student understanding that the same letter had to represent the same number. Then the second number sentence positioned the students to engage in argumentation in order to confront the misconception. After lengthy discussion the teacher recorded fifteen and fifteen as a possible solution and challenged the students with a question:

Teacher: Can J equal fifteen and B equal fifteen?
Zhou: Even if they are not the same letters they can still equal the same value… if it is a different letter it still could.

Reflective statements recorded by students after this lesson identified a change in thinking:

[21] $H + H = 30$ $J + B = 30$ What could H be? What could J be? What could B be?

Ruby: If it was the same letter it had to be the same number…I thought the different letters couldn't represent the same thing. Now I learned they can represent the same number.

Interview Data of Student Understanding of Variables Post-study

The final interview responses indicated that there had been a considerable shift in student use of notation to represent varying quantities. Table 2 illustrates the percentage of students correctly using notation in response to the questions[22].

Table 2
Percentage of Students (n=25) Using Notation for an Unknown Quantity

	Correct notation e.g., ■+ 3 $N + 3 + 2$	Non-standard or incorrect notation e.g., $B + 3 = A$, $A + A = C$	Number as notation e.g., 2 + 3 = 5	No response
Q. A	92%	0%	4%	4%
Q. B	92%		4%	4%
Q. C	44%	44%	8%	4%

Additional questions[23] were used with students working at a higher level (n = 19) which examined their understanding of what letters represented in a mathematical context. Significant improvement was demonstrated in their responses. In response to Question A, all of these students stated that the symbol in the equation could stand for six and justified this response. In response to Question B, 17 students stated that the symbol could stand for forty-five and justified this response. However, 1 student maintained the misconception that the variable could only represent a single digit.

A following question[24] was also used with these students to investigate their understanding of the concept that a different letter could represent the same number. Fifteen students identified that the number sentence was sometimes true and provided justification for their assertion.

Josie: Sometimes…because e and f could be the same numbers but then they could also be different numbers…b and b have to be the same number and n and n have to be the same number.

However, three of these students maintained that the same number could not be represented by two different letters in an equation.

Matthew: I don't think it's ever true because the first and the last letter are the same but the *e* and the *f* aren't the same and one number can't represent two letters.

In the final interview the improved student responses confirmed that the contextual tasks which provided opportunities to explore algebraic notation alongside specific pedagogical actions and extensive discussion had scaffolded student understanding of variables.

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[22] What is a mathematical statement or sentence to represent each of the following situations:
A) I have some lollies and then get five more.
B) I have some lollies, then I get five more and then I get three more.
C) I have some lollies then I get five more and then I double the number of lollies I have.

[23] $2m + 5$ A) Could the symbol stand for 6? B) Could the symbol stand for 45?

[24] Is $b + f + n = b + e + n$ always, sometimes or never true?

Conclusions and implications

This study sought to explore how younger students could be facilitated to represent their mathematical ideas using notation and gain a more sophisticated understanding of variables. Similar to the findings of MacGregor and Stacey (1997) many of the students initially based their interpretation of variables on intuition, guessing and false analogies. They also demonstrated misconceptions described by Knuth and his colleagues (2005).

Initial tasks involving algebrafied arithmetic problems provided students with a context to engage in discussion about variables. Both the task structure and specific pedagogical actions by the teacher including questioning and scaffolding to record appropriately supported students to begin using notation to represent a mathematical situation. Further opportunities for exploration of variables were provided through formalising functional rules. The teacher was able to utilise opportunities during these activities to shift students towards more formal notation. Results of this study also support Stephen's (2005) contention that carefully constructed mathematical tasks can be used to confront misconceptions about notation. Many of the students in this study deepened their understanding of variables. However, the small proportion of students who continued to demonstrate misconceptions about algebraic notation in the final interview reinforces the need for students to have multiple opportunities to explore symbolic variables.

Findings of this study affirm that younger students can begin developing their understanding about variables and be encouraged to use forms of notation. Carefully designed tasks, teacher intervention and extended discussion supported students to develop their understanding of notation. Due to the small size of this sample further research is required to validate the findings of this study.

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