Percentages: The Effect of Problem Structure, Number Complexity and Calculation Format

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This study reports how the difficulty of simple worded percentage problems is affected by the problem structure and the complexity of the numbers involved. We also investigate which methods students know. Results from 677 Year 8 and 9 students are reported. Overall the results indicate that more attention needs to be given to this important topic. Simple unit fraction equivalents seem to be emphasised, at the expense of fundamental definition ("out of a hundred") and apparently easy percentages such as 30%. The draft National Curriculum gives better guidance on the variation amongst percentage problems.

With a strong presence in the real world, percentage is a valuable topic in the school mathematics curriculum. Percentages are the method of choice for expressing fractional quantities in many real world applications, yet they are part of the mathematical world of proportional reasoning and so, as many studies have confirmed, they present challenges to learners (Cole & Weissenfluh, 1974; Parker & Leinhardt, 1995; Smart, 1980). This paper reports some findings related to Year 8 and 9 students' knowledge of this topic.

The questions examined in this paper are part of those being developed for an online diagnostic tool (www.smartvic.com) for probing the mathematics understanding of students. Descriptions are given by Price et al (2009) and Stacey et al (2009). The intention is that these 'smart tests' (Specific Mathematics Assessments that Reveal Thinking) will provide quick information to teachers on the extent and nature of understanding in their class and the prevalence of misconceptions about specific topics. Accompanying the results, which are available to teachers instantaneously, there is a description of the developmental stages, teaching suggestions and links to other resources. In addition to helping teachers with specific classes, the accompanying information is intended to increase the mathematical pedagogical content knowledge of the teachers using the tests. Note that the smart tests are unrelated to the author Smart(1980) above.

For most students, teaching about percentage spreads from the later years of primary school to Year 10. Percents are an everyday way of expressing fractional parts (e.g. I got 62% for Chemistry) with the advantage of easy comparison, and they are also the usual way of expressing multiplicative comparison and change (The price went up by 5%, this medicine reduces your chance of getting sick by 10%), relationships which are difficult to express otherwise. Four broad areas are assessed in the current version of the smart tests:

- **Relative size and equivalence:** use percentage to describe relative size (e.g. select which beaker is 68% full, describe a tree as about 50% taller than another) and link percents with fraction and decimal equivalents
- **Constant-whole calculations:** solve one-step problems of three types - find-part (e.g. 25% of 20 is x), find-percent (e.g. 5 of 20 is x%) and find-whole (e.g. 25% of x is 5)?

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- **Varying-whole calculations:** solve problems where the 'whole' varies (e.g. find that 10% of 20% of a quantity is 2% of it – first the initial quantity is the 'whole' and for in the second part 20% of it is the temporary 'whole')
- **Adding percent is multiplying:** identify adding a percent with multiplying (e.g. add 3% by multiplying by 1.03; repeated adding of 3% is multiplying by powers of 1.03)

The Victorian Essential Learning Standards (2008) for the compulsory years of school specifically mentions percent at only a few points. The Number dimension specifies "relative size and equivalence" ideas: finding equivalences such as ¾ = 0.75 = 75% = 3 : 4 at level 4 and extends to ratios such as 2 : 3 = 4 : 6 = 40% : 60% (sic) at level 5. Two other dimensions make mention of percentage, both of which are 'find-percent' problems from the constant-whole calculations above. A supplementary part of VELS (Progression Point Examples in Mathematics) includes work from all four areas above and therefore goes further than VELS, presumably with appreciation of the many real world problems that relate to percentage. Textbooks also generally go further than VELS. A survey of six current textbooks showed that all covered find-part and find-percent problems, and half of the Year 8 textbooks included find-whole problems, sometimes as extension work and mainly done by the unitary method, written as a series of separate steps.

This paper gives the results from a study trialling the percentage items. The main aim is to report on the degree of mastery shown by middle secondary students and how this is affected by problem structure, number complexity and calculation format (defined below). We report only on the second of the four broad areas above (constant-whole calculations).

Literature Review

Ashlock, Johnston, Wilson, and Jones (1983) cited in Dole, Cooper, Baturo, and Conoplia (1997) observed that there are three different problem structures amongst the constant-whole percent situations, which we label 'find-part', 'find-percent' and 'find-whole' (examples above). Dole et al (1997), reporting a study of 18 students in Years 8, 9 and 10 after screening 90 students, classified them at three levels. Their classification system implies that find-part problems are easier than the other two problem structures. In this study, we will report on the relative difficulty of the three problem structures.

Dole et al (1997) found that only students able to solve all three types could identify these different types of percentage questions, analyse them in terms of their meaning and predict the operation to be used as well as the size of the answers relative to the other numbers given. Students only able to solve find-part problems were reliant on formulas and, if they forgot them, resorted to trial and error. They were able to predict the value of the answer but unable to construct strategies to calculate them.

Koay (1998) investigated the understanding of percentages of pre-service primary teachers in two Singapore teacher education courses. The "find-part" problem (find 75% of 160) was done correctly by 98% of the 224 students, with about 90% of them showing a fraction calculation. The "find-percent" problem (what % is 7 of 28?) was of intermediate difficulty and the most common error was to calculate 7/100 × 28. The find-whole problem (40 is 80% of what number?) was the most difficult with a facility (percent correct) of 85%. No students used a fraction or decimal calculation (dividing by 0.8 or by 80/100) but instead they used unitary, ratio or algebra methods. Koay concluded that knowledge of percent was often rigid and rule-bound, even amongst pre-service teachers. She recommended that greater understanding of percent would arise from teaching using better visual models and using more real life examples. Significantly, more than a third of the students were unable to give a second method to solve the problem. In this study, we report on the calculation methods that students are able to use to solve percentage problems.

As well as the ubiquitous difficulties with proportional reasoning, some specific features affect the difficulty of percent problems. VELS emphasises common fractions and known benchmarks, which indicates that number type is a factor and this will be tested in this study. Another factor affecting the difficulty of "find the whole" problems especially is the well known and persistent 'MMBDMS' (Multiplication Makes Bigger, Division Makes Smaller) misconception (Bell, Swan & Taylor, 1981). Many students will choose to divide rather than multiply (and vice versa) in problems where the divisor or multiplier is a number between 0 and 1. This is because students associate multiplication with making numbers larger and division with making smaller. In our study students are required to identify which calculations could compute the whole when 11% is 145. The MMBDMS misconception could cause them to reject 145 ÷ 0.11. Possibly in order to avoid the MMBDMS misconception, three of the six surveyed textbooks did not find-whole by division of the decimal or fraction, but used the unitary method.

There are several sources of data on students' success with percent problems of the constant-whole type. Ryan and Williams (1997) found that 38% of 14 year olds were able to compute the percentage remaining when 60 books were taken from 80. 31% of students gave the answer 20, rather than 20%. This "% sign ignored" misconception was common across contexts. The facility for a variety of find-part and find-percent problems varied from 15% and 49%. "Percent means divide" was another misconception, when, for example, students calculate 24 out of 500 by finding 500 ÷ 24.

Method

Participants and Procedure

The data reported in this paper comes from four secondary schools in Victoria, representing a range of socio-economic backgrounds. The testing took place in Term 3 of 2008; at this time, some Year 7 students and all Year 8 students from the four schools were tested and all Year 9 students at three schools. Only students who returned consent forms (57% over the whole project) are included in this analysis. The normal classroom teacher took their complete class into a computer laboratory for the duration of a normal lesson for the test. An on-screen calculator was available. There were many items in the tests across all strands of the mathematics curriculum, and three different test versions were used, to reduce copying, and to vary the order of items. This data was being collected to trial and calibrate the smart tests. From this data, we have extracted two large subgroups to report on percentage: 342 Year 8 students who answered at least one question of the three multi-question items labelled #363, #370 and #377 and 335 Year 9 students who answered at least one question of the three multi-question items labelled #62, #63, #64.

Problem Structure and Number Complexity

The Year 9 students completed constructed response items covering the three problem structures above; find-part (#62), find-percent (#63) and find-whole (#64). These three items were provided as word problems with multiple questions, each question having a different number complexity. The levels of number complexity used were *definition* – relying only on an understanding that percent means "out of a hundred"; *simple* – using percentages equivalent to unit fraction with denominator a small factor of 100 (e.g. 2, 4, 5, 10); *medium* – using percentages with well known non-unit fraction equivalents (e.g. denominators of 5, 10, 8); and *hard* – using difficult numbers so that a calculator is required to produce an answer. For example, the question: *A bookcase has 100 books. 7% are picture-books. How many picture-books are there?* (#62A) is a *find-part* question with *definition* level number complexity. Table 1 summarises the items.

Table 1
Summary of Problem Structure and Number Complexity (#62, #63 and #64)

Number Complexity	Problem Structure		
	Find-part #62	Find-percent #63	Find-whole #64
A) Definition	7% of 100	29 out of 100	1% is 3
B) Simple	25% of 40	15 out of 30	10% is 4
C) Medium	30% of 60	35 out of 40	80% is 12
D) Hard	13.5% of 1294	78.85 out of 95	87% is 163

Problem Structure and Calculation Format

The Year 8 students completed three multiple-question, multiple choice items which examined the types of calculations that they recognise as solving a percentage problem (#363, #370, #377). All of these items are of *hard* number complexity. Since calculation methods depend to a large extent on the type of number being used, the calculation formats are classified as fraction, decimal and whole number.

Figure 1 shows the find-whole item #370 (145 is 11% of x). Part (a) is a multiple choice estimation question designed to prompt students to think carefully about the problem. This estimation was not included in the other problem structure items (#363, #377). In part (b) students were required to select one of the multiple choices (*right*, *wrong* or *not sure*) from a drop down box for each of the six calculations provided. Note that they did not need to actually perform any of these calculations. These six calculations have been created in three pairs (one right and one wrong calculation) in three different formats.

a) Tom's class won 145 lollies at the quiz. This was 11% of the lollies. The total number of lollies at the quiz is: (a) Less than 145 (b) About 145 (c) About 300 (d) More than 1000	b) Which of these calculations will find how many lollies at the quiz? i) 145×0.11 ii) $145 \div 0.11$ iii) $(145 \div 11) \times 100$ iv) $^{11}/_{100} \times {}^{145}/_{1}$ v) $145 - 11$ vi) $^{145}/_{11} \times {}^{100}/_{1}$

Figure 1. Item #370 (find-whole), showing a) estimation question and b) calculation format questions.

We use the terms *right / wrong* to describe questions (so (i) is a wrong calculation) and *correct / incorrect* to describe students (so a student who rejects (i) is correct on that question). Calculation (iii) is a right calculation and a student who accepts it is correct, whereas a student who rejects it is incorrect. Calculation (iii) reflects the thinking behind the unitary method. Note that decimal calculation format is logically impossible for find-percent calculations. The complete set of items is given later in Figure 3 with the results. Note that all fraction formatting in the test was done correctly with multi-line formatting.

Results and Discussion

Problem Structure and Number Complexity (items #62, #63 and #64)

Figure 2 provides the facility (percent correct) for Year 9 students on items investigating the effect of problem structure and number complexity. The facilities are generally in the order find-part (#62) (which Dole et al (1997) and Koay (1998) found easiest) then find-percent (#63) then find-whole (#64). With the exception of #62B (with answer 50%), as the number complexity increases, the facilities decrease. The broad pattern is consistent with previous studies such as Koay (1998).

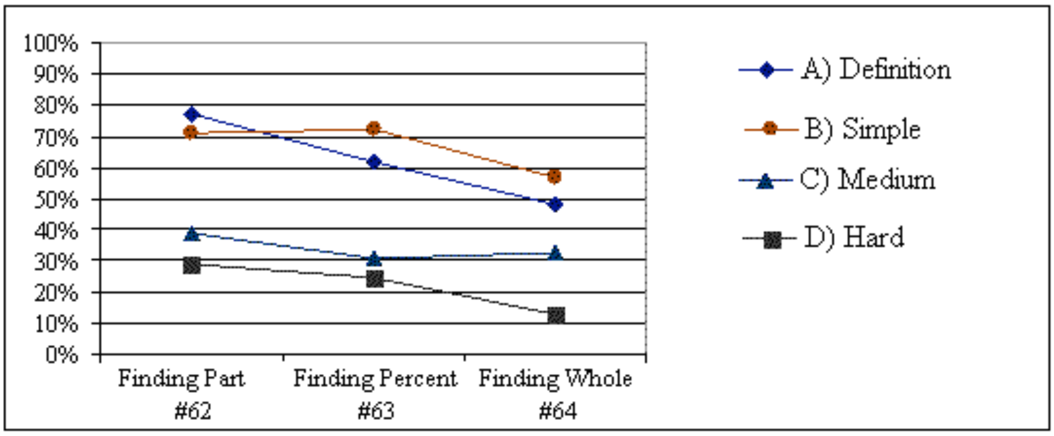

Figure 2. Facility of 335 Year 9 students on problem structure vs number complexity questions.

We discuss several features. First the facilities are very much lower than desirable for a topic central to basic numeracy. With very simple, highly readable, worded contexts, only 62% of students knew that 29 out of 100 is 29% (#63A) and only 39% could find 30% of 60 (#62C). The other feature is that, broadly speaking, the results show that questions with *definition* and *simple* number complexity are of similar difficulty (the top two lines in Figure 2). The similarity of facility for the *definition* and *simple* questions raises the possibility that instruction is emphasising the simple fraction equivalents (half, quarter) even more than the fact that percent is "out of a hundred". This possibility is consistent with our reading of VELS.

The similarity of facility of *medium* and *hard* number complexity questions demonstrates that students regard only a small range of fractions (and equivalents) as familiar. For example, we had expected that finding 30% of 60 could be handled by many students mentally (10% is 6, so 30% is 18) but the facility (39%) of this question is very close to the facility (29%) for of the difficult number question. Given the usefulness of being able to carry out calculations such as 30% of 60 mentally, this result is of concern.

Problem Structure and Calculation Format (items #363, #370, #377)

Figure 3 shows graphically the results for the questions where the Year 8 students (N = 342) had to decide if given calculation methods were right. Students were not asked to do the calculation, although some may have checked calculations by doing them. The percentage of responses that were blank was similar across all questions (range of 5% to 10%), so the reasons for not answering seem not greatly affected by question type or difficulty. The percentage of responses that were ‘not sure’ were reasonably similar across all questions (range of 6% to 19%), although (as may be expected) there is a moderate trend (r = 0.45) for more students to be unsure on questions with lower facility, which indicates that the option of choosing “not sure” is being used as intended. The find-part questions were best done (average facility of 54%) and the find-percent and find-whole questions had equal average facility (44%), but this reflects both the range of choices offered as well as the problem structure. The wrong choices 145 – 11 and 87 – 19, for example, were easy to reject and were only offered in the find-percent and find-whole categories. As can be seen from patterns in Figure 3, students were more frequently correct with (rejecting) wrong calculations (54% on average) than with (accepting) right calculations (36% on average). The average facility on the three calculation formats (decimal, fraction, whole number) was approximately the same (45% - 50%).

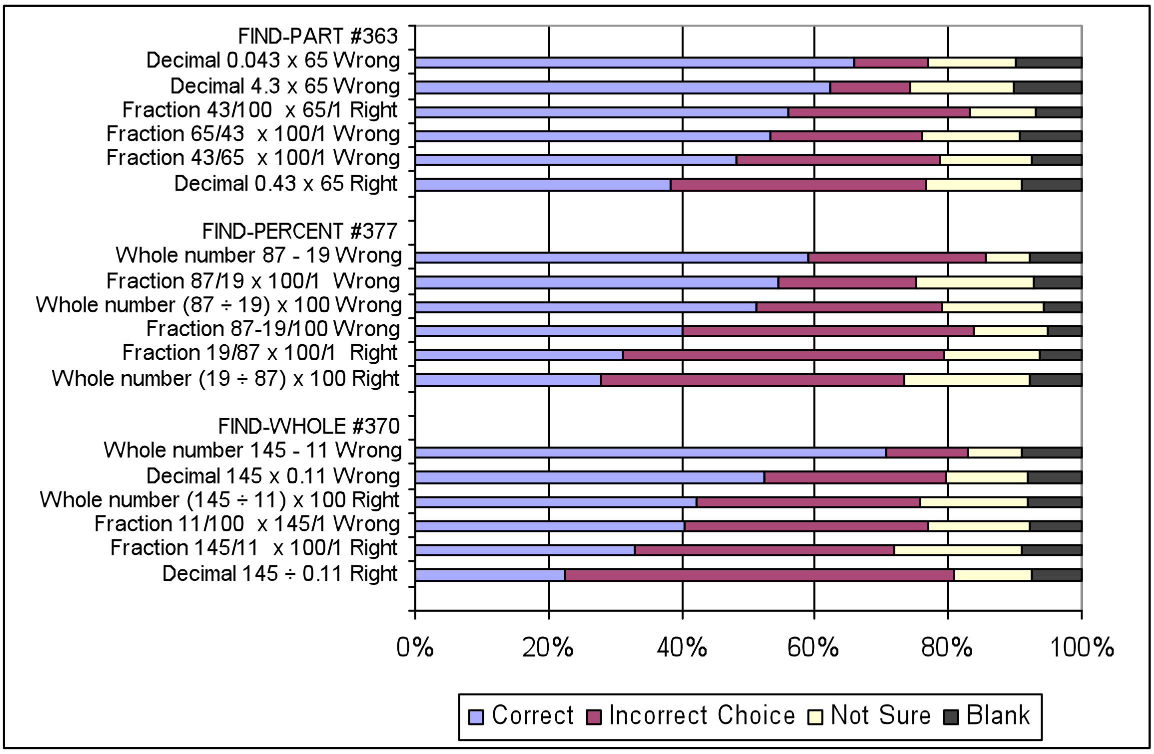

Figure 3. Percent of 342 Year 8 students giving each response by calculation formats.

Looking more closely at individual questions in the find-part item (#363) shows that the right fraction calculation ($^{43}/_{65} \times {}^{100}/_{1}$) was accepted relatively well (56% correct) and considerably better than the right decimal calculation with only 38% accepting 0.43 × 65. This is also the case for the other comparison of right decimal and fraction calculations (find-whole). It is somewhat surprising that 30 years after “five dollar” four function calculators became available to everyone in Australia it appears that fraction methods for percent questions still dominate over decimal methods in schools. This is confirmed in the textbooks.

The most difficult question was the right, find-whole, decimal question (145 ÷ 0.11), which involves division by a number less than 1 to give a larger answer. It is likely that the MMBDMS misconception makes this question especially difficult. Division by a number less than 1 is not evident in other calculation formats. The right, find-whole, whole number calculation (145 ÷ 11) x 100 was relatively very well done; this may be because it reflects the steps of the unitary method. Our modest survey of Year 8 textbooks indicated that the unitary method was the method most commonly taught for find-whole questions, but the method is taught as a series of steps, rather than as one calculation, so students have done well to recognise it in the compound calculation offered in the item.

Koay's observation that many of her students did not know more than one method for solving a percentage problem is also probably influencing the data in these questions, although the effect cannot be measured. Students who believe there is only one way to do a calculation may reject others without examination after finding one that is right. Even though the choice "right / wrong / unsure" was placed alongside each question, it is also possible that a few students thought that there would be only one right answer in the batch of calculations as in a typical multiple choice question.

Finally, it is also interesting to compare the accuracy of the estimation in item #370 (see Figure 1) with the calculations chosen. The correct answer (over 1000) was chosen by 56% of the Year 8 students, so that it is easier than nearly all of the calculation format questions. The group of students who estimated correctly was no better at selecting the calculations correctly. For example, 15% of correct estimators accepted the right decimal calculation (145 ÷ 0.11) and rejected the wrong one (145 × 0.11) compared to 17% of incorrect estimators. This is consistent with Dole et al (1997) who found most students were able to predict the expected answer of a percentage problem but did not or could not use this to guide their calculations.

Implications and Conclusion

As noted in the discussion above, the facilities of the percent items in almost all categories were low. The analysis of the students who omitted answers indicates that students took the test seriously. The problems were straightforward, with simple wording, clear layout and no extra data. The results are concerning, given the real world importance of percent. Moreover, the problems reported in this paper are only in the second of the four areas of percentage problems that should be part of the curriculum. Problems from the (harder) third and fourth types are also of importance in the business world. Both problem structure and number complexity were contributed to difficulty. The difficulty that students had with problems involving 100% directly (definition-level complexity) and with medium-level percentages (e.g. 30%) was very surprising. The fraction method for the find-part calculation (${}^{43}/_{65} \times {}^{100}/_{1}$) was the only right method that was correctly recognised by over 50% of students. Right decimal methods were generally not recognised. This probably reflects the continuing dominance of fraction methods in textbooks, which is surprising given that they are not so simple to implement on a calculator or spreadsheet.

To improve the teaching of percentage may need more emphasis in the curriculum (see below), better teaching methods and a review of the methods being taught. Although we have no hard evidence, our experience is that it is very effective to use the dual number line (DEECD, no date) as a graphic organiser for (nearly) all types of proportional reasoning problems, including the constant-whole percentage problems. Additionally, it may be worthwhile to review the methods being taught to solve percent problems. For example rather than teaching separate methods for each problem structure, students could start with a formula that applies across problem structures.

In our opinion, the VELS (2008) statements do not describe an adequate level of mastery of percent. Finding parts, percents and 'the whole' are all common problems, encountered in the course of everyday and business life. A common find-whole problem, for example, is to find the cost before GST of something which costs $25 after 10% GST has been added. All four areas of percentage knowledge as outlined in the introduction and certainly all three problem structures for the constant-whole calculations should be part of a rounded mathematics curriculum. The draft National Curriculum (ACARA, 2010) has better coverage than VELS, specifically including in the elaborations find-part, find-percent and find-whole problems, as well as varying-percent calculations and examples (such as compound interest and population decay) which require knowledge of the fourth part of percentage knowledge, that adding a percent is the same as multiplying. We hope that clearer expectations may assist in giving students the opportunity to develop better knowledge of this important part of mathematical literacy.

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