Rates of Change and an Iterative Conception of Quadratics [1]

Karoline Afamasaga-Fuata'i
University of New England, Australia
<kafamasa@pobox.une.edu.au>

Investigating students' conceptions of covariation patterns between quantities situated within contextual settings engenders enriched, deep understandings of functional relationships. This paper presents data from a case study of a student (Mary) who solved quadratic contextual problems. Mary's schemes, constructed from quadratically related quantities and patterns of additive rates, fostered the development of an iterative, summative conceptualisation of quadratics in contrast to the product view. Findings support the use of contextual problems to motivate students to think reflectively and mathematically.

As educators we are perpetually concerned with determining and developing better ways of presenting mathematics to students to promote deeper conceptual understandings. The constructivist's perspective encourages educators to focus on and listen to what students have to say and do when solving problems. Various *Curriculum Standards* (NCTM, 1989, 2000; NSW, 2002) exhort that students explore patterns and functional relationships in realistic situations and communicate their mathematical understanding and models effectively using multiple representations. This paper focuses on Mary's strategies in solving a maximum quadratic *contextual* problem in terms of her *interpretations* of the situation, actions taken to resolve her problematics (i.e., conceptual obstacles) and multiple *representations.* Whilst struggling with competing interpretations and representations, she developed potentially useful schemes reflectively abstracted from explorations of numerical patterns of tabular data. The theoretical framework and methodology of the overall study in which Mary was one of four participants is briefly outlined followed by the data, a brief discussion and some conclusions based on Mary's first two sessions.

Theoretical Framework

The theoretical framework is based on Piaget's epistemology in which learners actively organize their experiences by constructing *schemes* to assimilate and/or accommodate new knowledge. Constructivists view mathematics as a human creation that historically evolves within cultural contexts through social interactions, reflection, communication and negotiation of meanings. Humans construct mathematical concepts to structure experience and to solve problems. (Confrey, 1991a, 1991b, 1994; Confrey & Doerr, 1996). Accordingly, mathematics can be created as a result of students' actions in situations. Through reflective abstraction on their actions (i.e. abstraction of the relationship between actions and effects of those actions), students construct schemes, modify and/or apply them intentionally to achieve their goals. (Confrey, 1994; Confrey & Smith, 1994; Steffe, 1994; Hershkowitz, Schwarz & Dreyfus, 2001). When solving problems, students begin by identifying their problematics, acting on them and then reflecting on the results of those actions to create operations, followed by checks to determine whether problematics are resolved satisfactorily. The cyclic activities: *problematic→actions→reflections,* therefore

[1] This study was part of the author's doctoral dissertation supervised by Dr Jere Confrey at Cornell University. Some data reported here formed part of a presentation at the Multiliteracies and K-12 English Teaching Conference held at UNE on November 26, 2004 to illustrate the importance of critical literacies in mathematics.

consists of an anticipation, action and reflection. If proven successful, it is repeated in other circumstances to create a “scheme,” a more automated response to a situation (Confrey, 1991b, p. 120). Over time, these schemes emerge from assimilations of experience to ways of knowing, have duration and repetition, and are more easily examinable than isolated actions (Confrey, 1994, p. 320). Assimilating an object into a scheme simultaneously satisfies a need and confers on an action a cognitive structure (Thompson, 1994, p. 182). Tasks are selected for their potential to invite and motivate students to engage with the mathematical idea and should yield to multiple interpretations and resulting approaches (Confrey, 1991a, 1994; Confrey & Doerr, 1996), hence the use of *contextual problems* (problems with realistic *contexts*). Multiliteracies in mathematics include the requisite critical skills to interpret the mathematics embedded in various representations such as the numerical, symbolic, algebraic, and graphical. Solving contextual problems therefore effectively demands that students have the multiliteracy skills to decode the question in order to respond appropriately and ability to critically appraise the problem context so that relevant, embedded mathematical tasks are identified (Zevenbergen, Dole & Wright, 2004) as well as shift flexibly between different representations. By listening to students, we learn from them; our mathematics understandings are challenged and enriched as a result of it. Hence, analyzing student data can prompt the *re-examination and extension* of one’s mathematical understanding to new territories of mathematical meanings (Confrey & Smith, 1994, p. 136).

Functions play a central and unifying role in school mathematics (Confrey, 1991a; Romberg, Carpenter, & Fennema, 1993, NCTM, 1989; Knuth, 2000). The *2000 NCTM Standards* recommend that students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge, and consider the study of patterns and functions as one of its central themes. The literature on *functions* describes two views. The *correspondence view* is more consistent with the modern set-theoretic formal definition associated with Dirichlet-Bourbaki, and the *covariational view* is reflective of the historical development of functions and consistent with Euler’s classical view (Smith & Confrey, 1994, p. 335). Prevalent in school mathematics is the correspondence view. Students primarily work with algebraic forms (analytic expressions) of functions as correspondence rules. Their understanding is predominantly built from using these algebraic expressions as algorithmic procedures that take inputs to generate corresponding output values (Confrey & Smith, 1994; Smith & Confrey, 1994; Knuth, 2000). In contrast, a more intuitive approach is the covariation view. Students observe and reflect upon covarying patterns with quantities in tabular form; describe patterns in one column in relation to values in another and construct *rates of changes* in terms of repeated action in each column (Confrey & Doerr, 1996; Smith & Confrey, 1994). Coordinating multiple representations and reflecting upon their actions, students construct viable schemes thus engendering a more natural view of algebraic symbolism as the need to codify actions and operations that one takes to describe variations of quantities, rather than an abstract, symbolic system devoid of contextual origins (Confrey & Smith, 1994; Smith & Confrey, 1994). It also gives students ownership of the mathematics they construct.

The literature presents multiple perspectives on rates of change and the related concept of ratio (see Thompson, 1994). The most relevant one to this paper is that by Confrey and Smith (1994) whose situational view is similar to Behr, Lesh, Post and Silver (1983) and others. However, Confrey and Smith expand it further by proposing a more grounded approach to “rate” that emphasizes and values both quantities that are being compared (numerator *and* denominator), and indicates the kinds of mental actions that are applied to the quantities. For example, *ratio* is defined as the "invariance across a set of equivalent proportions" while *rate* is defined as "*a unit per unit comparison*". Conceptualising rate as an intentional, coordinated and repeated action between two quantities recognises that rate can be constant (additively or multiplicatively) or varying whilst ratio remains constant. This view "allows one to explain uniformity of unit to unit comparisons (homogeneity) and the variation in rates over time (non-homogeneity)" (Confrey & Smith, 1994, p. 153-154). The case study reported here investigated in-depth the development of students' schemes constructed from representations and abstractions of numerical patterns of additive rates of change to model quadratic contextual situations (see Afamasaga-Fuata'i, 1991 for details). Additive rates follow an arithmetic progression such as those for polynomials whilst multiplicative rates are geometric progressions such as with rates of exponential functions.

Methodology

The research methodology was a constructivist teaching experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Duit & Confrey, 1996; Steffe & D'Ambrosio, 1996; Duit, Treagust, & Mansfield, 1996). It was set out to model students' developmental understandings of mathematical concepts. The inclusion of realistic contexts was deliberate to provide critical sites for students' mathematizing activities (Confrey, 1994, 1991a; NCTM, 1989) and to foster the construction of quadratic schemes. Students were encouraged to use Function Probe® (FP) (Confrey, 1991c), a multiple representational software, to support their activities when making, confirming and revising conjectures. Of particular importance was FP's use to automatically generate table values (i.e. x, y, Δx, or Δy) with the input of an equation to fill columns, or manually one by one where the symbol Δ denotes differences between consecutive values. Individual interviews were interactive and centred around a set of problems. The researcher would ask probing questions to gain insights into a student's constructions, interpretations and reasoning processes (Cobb et al, 2003; Confrey, 1994; Duit & Confrey, 1996; Confrey & Doerr, 1996; Thompson, 1994) whilst simultaneously promoting more self-reflection and a stronger approach to knowledge construction for the student (Confrey, 1991a). The nature of probing questions depends on student responses, need for clarifications and justifications, and/or potential pathways that arise which promise greater insight into powerful ways of thinking (Thompson, 1994, p.195). It is acknowledged that communicating one's rationale and reasoning processes to another simultaneously shapes and transforms one's reflective thinking and schemes of internalised actions (Confrey, 1994, p. 321). All problem-solving sessions were videotaped; sessions were approximately two-hours each, and held three-times weekly for at least 6-weeks. Resources available included pen-and-paper and the software FP. A pre-test was conducted to select four students who demonstrated a solid understanding of linear functions. Mary was an American, second year university arts student at the time of the study with a high school general mathematics background. Data collected included students' worksheets and FP files, researcher's notes, and transcripts of each session. The starting point is the familiarization tutorial where students learn to use FP whilst solving a linear contextual problem. They investigate distances of n^{th} posts from a house with a gate of width 4-feet attached to it on one side while the opposite side has the first post (n = 1) with other posts spaced 3-feet apart (to be referenced the gate-problem). The distance-equation $d = 3(n - 1) + 4$ is given to students.

Data and Analysis

Mary's cycles of activities are described below as she developed a robust conceptualisation consistent with her preferred interpretation. To distinguish between the researcher's interpretations of Mary's actions (in normal font) and Mary's own words and symbols, the latter will be "*italicised*" and enclosed in quotes.

Multiple Interpretations and Representations

The first contextual problem is as shown:

> Farmer Joe has records showing that if 25 avocado trees are planted, then each tree yields 500 avocadoes (on the average). For each additional tree planted, the yield decreases by 10 avocadoes per tree. Determine the number of trees that would maximize total yield.

Mary began by proposing various literal interpretations and multiple representations. Her initial conceptualization as the sum of "*yield from 25 trees + yield from additional trees*" was represented as "*total yield = 500 + yield from additional trees*" apparently conceived to be structurally similar to the distance-equation. It was also inconsistent with the phrase each tree yields 500 avocadoes (on the average) in the problem statement. In responding to probes, Mary referred to her linear schemes in which she connected the: (a) "*number 3 in front of the (n - 1) term*" of the distance-equation to the tabular "*constant difference (Δd = 3)*" of the FP column and contextual condition of "*3-feet equal spacing*", and (b) "*constant 4*" to the contextual condition of "*fixed gate-width of 4-feet*". Mary explained: "*I'm trying to relate this (new problem) … I'm trying to find some kind of relationship so that it's easier because I understand this (gate-problem) … it's the same thing ... because … we have a starting point*". Whilst critically conceptualising total yield, Mary was simultaneously cognizant of two things; that the amounts of 500 and 25 gave an "*initial amount of 12500*" and her first conceptualisation requires modification to accurately represent her interpretation "*increasing consecutive total yields, but with differences that are decreasing by 10 such as from 490 to 480 to 470*". Subsequently, the revision became "*500(n + 25) – 10(n + 25) = total yield*" where "*n*" is additional trees. However, numerical evaluations showed "*constant differences of 490*" not the expected "*sequence of 490, 480, 470*". Modifying once again her expression to "*500(n + 25) – 10(n) = total yield*" produced total yields "*12500, 12990, 13480, 13970*" not the expected values "*12500, 12990, 13470, 13940*". In terms of Thompson's (1994) distinction between quantitative and numerical operations (p. 184-188), Mary quantitatively conceived total yield as the sum of two quantities, an initial amount and yields from additional trees, perhaps structurally similar to the distance-equation, but has yet to fully conceptualise the required numerical operations to construct the new quantity "*yield from additional trees*". After much systematic modification and evaluation, Mary verbally re-affirmed the salient initial conditions and re-presented them on her worksheet as shown in Figure 1a.

Strategically, Mary was recapitulating, reassessing, and reflectively thinking through her actions and multiple representations thus far as she contemplated a way forward. Whilst setting out her expected values in tabular form (see Figure 1b), Mary experienced a significant moment of insight (Barnes, 2000) – the "*constant difference of 10*" was similar to the "*constant difference pattern of 3*" in the gate-problem. This immediately signalled the opportunity to apply her linear schemes. Hence, after experimenting with, and evaluating various expressions, Mary derived the expression "*(500 - 10n)*" labelled "*fruits gained*" to illustrate the "*starting average yield of 500 and decreasing yield of 10*", see

Figure 1c. For verification, Mary generated "*y = 500 – 10n*" values as shown in Figure 1d where "*y*" represented "*yield per added tree*".

1. 25 trees = 500 fruits/tree

(a)

Fruits Gained = 500 – 10n

Distance d = 3(n – 1) + 4

(c)

	Yield of additional trees	difference
25	500	
26	490	
	10	

(b)

n	y=500-10n	Δy
additional trees	yield per added tree	
1.00	490.00	-10.00
2.00	480.00	-10.00
3.00	470.00	-10.00
4.00	460.00	-10.00
5.00	450.00	-10.00

(d)

Figure 1. Mary's multiple representations of yield per additional tree.

Returning to her earlier total–yield problematic, she explained: "*I'm trying to think of a way to … create a column in which … it will show the accumulation ... the cumulative … including the 25*". Acknowledging the inadequacy of "*(25 x 500) + (500 – 10n)*" to account for "*cumulative totals*", she declared she wanted a "*summation of 500 – 10n*" to represent a "*running total*" from all additional trees and a formula for her operational scheme of iteratively adding new yield to previous total yield as illustrated by her procedural actions of "***12500***, *12500 + 490 =* ***12990***, *12990 + 480 =* ***13470***, *13470 + 470 =* ***13940***". Mary represented this as "*Total Yield = 12500 + ∑(500 – 10n)*" where "∑" signified her "*cumulative running total*" of (500 – 10n) values. While thinking aloud, Mary generated t values by typing into FP formula "*t = 12500 + (500 – 10n)*" and "*Δt*" values using the Δ-menu command. A comparison of generated t values and handwritten ones on her worksheet (the latter are shown as columns a and Δa in Figure 2a) showed a mismatch of Δt and Δa values. After further unsuccessful experimentation, she used the letter "*t*" to represent "*Total yield*" and "*s*" for cumulative yields "*∑(500 – 10n)*" and subdivided equation "*Total Yield = 12500 + ∑(500 – 10n)*" in two parts as shown in Figure 2b. Her ensuing dialectic among reflectively abstracting features of various algebraic representations (for s such as "*s = n*y = n*(500-10n)*"; "*s = ny + 10n*" and "*s = ny – 10n*") and tabular data (columns "*b*" and "*s*" in Figure 2c) was mediated by numeric evaluations of differences as shown in column "*c = b – s*"; c values quantitatively represented the increasing shortfall of equation "*s = n(500 – 10n)*" to match expected "*b values*". Mary used the symbol "*" to reinforce her strong belief that quantity s must be "*n times another quantity*" which was yet to be fully developed.

To encourage an alternative approach, the interviewer asked: "Assuming that all trees were affected by the addition of new trees, how would that change your formula for total yield?" The response was immediate "*Total Yield c = (n + 25)(500 - 10n)*" where "*c*" represented the other interpretation. However, this was dismissed temporarily as it was not her preferred interpretation. Notwithstanding that, Mary re-considered, then argued that choosing equation c means quantitatively "*c values are less than expected t values*", see Figure 2d. Illustrating further for n = 2, "*Δt = 480 compared to Δc = 220*"; she pointed out

“*Δc was not accounting for 260 of the fruits*” (i.e. “*Δt - Δc = 260*)”. Extending her plausible reasoning (Lithner, 2000) forward from 27 (n = 2) to 28 (n = 3) trees, and backward to 25 (n = 0) and 26 (n = 1) trees, Mary’s numeric analysis confirmed “*unaccounted for fruits*” (q = Δt - Δc) were increasing by a “*constant difference of 10 from 250 each time a tree was added*” (columns n, q and Δq in Figure 2d). Subsequently, she conjectured that “*t = c + ∑unaccounted for fruits*”. Evidently, Mary attempted to: (i) develop an algebraic bridge between equations “*c = (n + 25)(500 – 10n)*” and “*t = 12500 + ∑(500 – 10n)*”, and (ii) explore the potential value of Δq (Δq = Δ(k – p) = Δ(Δt - Δc)). However, Mary finished off the first session having established that “*unaccounted for fruits*” were increasing by 10, but had yet to algebraically represent it.

Table

n	y=500-10n	t	Δt	a	Δa
additional					
1.00	490.00	12990.00	-10.00	12990.00	480.00
2.00	480.00	12980.00	-10.00	13470.00	470.00
3.00	470.00	12970.00	-10.00	13940.00	460.00
4.00	460.00	12960.00	[illegible]	14400.00	

t has equation t = 12500 + (500 -10n)

a is expected total yields with cumulative total for y

(a)

t	=	12500	+	s
		first part		second part
Yield from:		25 initial trees		additional trees

(b)

n	y=500-10n	b	s=n(500-10n)	c=b-s
Additional trees	Yield per added tree	Expected cumulative yields from additional trees	Predicted cumulative yields from additional trees	Difference between Expected & Predicted cumulative yields
1.00	490.00	490.00	490.00	0.00
2.00	480.00	970.00	960.00	10.00
3.00	470.00	1440.00	1410.00	30.00
4.00	460.00	1900.00	1840.00	60.00

(c)

n	t	k=Δt	c	p=Δc	q=k-p	Δq
additional trees	Expected total yields according to Mary's preferred interpretation	difference between total yields	Expected total yields if all trees had the same average yield	Difference between total yields	Mary's unaccounted for fruits	difference between unaccounted for fruits
0.00	12500.00	490.00	12500.00	240.00	250.00	10.00
1.00	12990.00	480.00	12740.00	220.00	260.00	10.00
2.00	13470.00	470.00	12960.00	200.00	270.00	10.00
3.00	13940.00	460.00	13160.00	180.00	280.00	10.00
4.00	14400.00	450.00	13340.00	160.00	290.00	
5.00	14850.00		13500.00			

(d)

Figure 2. Multiple representations of total yields.

In the subsequent session, Mary finalized her interpretation and algebraic representation to be *c* because: “*it will make sense ... the more trees you plant … the less nutrients there are in the ground … and the less fruits produced by the tree*”. She further recognized that the c was much easier to work with than t because “*coming up with the cumulative formula is easier*”. Determining maximum total yield (of 14060 at both 37 and 38 trees) was a matter of extending FP generated values. Most importantly overall, Mary made significant progress in algebraically and numerically representing her preferred interpretation *t* in terms of: (a) an initial amount and summation “*t = 12500 + ∑(500 – 10n)*, (b) a known function and summation “*t = c + ∑unaccounted for fruits*”, and (c) an iterative and summative operational scheme. She also identified two Δ-variation patterns; those that are constant (Δd in gate-problem, Δt in Figure 2a, and Δq in Figure 2d) and those that vary (Δa in Figure 2a, k = Δt and p = Δc in Figure 2d). Her linear schemes were successfully applied in deriving yield per tree “*(500 – 10n)*”. Mary repeatedly used the expression “*unaccounted for fruits*” as a label to refer to the quantity “*Δt – Δc*” (q in Figure 2d) to argue for her preferred interpretation and was surprised that “*differences of Δt – Δc*” (i.e. Δq) were constant.

Conclusions and Recommendations

Conclusions from Mary's first two sessions on the avocado problem are organized around three main themes: iterative conception of quadratics, unit comparisons of rates and construction of schemes. *Iterative Conception of Quadratics* - Mary's concept of "*unaccounted for fruits*" prompted a post-interview re-conceptualisation of quadratics primarily to unpack the product view iteratively. In contrast to Mary's initial interpretation of perceiving loss only from the latest additional tree, the product view "*c*" interpreted loss to be from all trees. This was the main conceptual difference Mary tried to reconcile algebraically, procedurally and quantitatively in the t⇔c bridge equation. Iteratively, c is "Total Yield = Previous yield + Yield from new tree – Loss from new tree – Loss of 10 more per old tree", and generalized algebraically as $c = 12500 + \sum(500 - 10n) - \sum 10((n + 25) - 1)$. Clearly, the "*unaccounted for fruits*" Mary was seeking algebraically is another summation $\sum 10((n + 25) - 1)$. Conceptually and practically, it represents the distributed loss of 10 more per tree from old trees as a result of the latest additional tree. Most probably, with more time in the first session, Mary would have derived the algebraic expression $(10(n + 25) - 1)$ by applying her linear schemes to the sequence 250, 260, 270, …. But beyond that, Mary would be faced yet again with another summation problematic. In spite of it, her initial preferred iterative and summative view of quadratics provided a viable contrast to the product view. *Unit Comparisons of Rates* - Mary's struggles to develop appropriate formulas to represent preferred interpretations continuously led her to explicitly examine rates of changes as unit per unit comparisons. By applying linear schemes premised on constant additive rates of change and initial amounts, she constructed the expression "*(500 - 10n)*". The quantity "*unaccounted for fruits*" was mediated meaningfully by numeric analyses of total yield values (t⇔c) and rates of changes ($\Delta t \rightarrow \Delta c \rightarrow \Delta q$) where Δt and Δc exemplified varying additive rates of change. In so doing, she spontaneously encountered differences of differences ($\Delta q = \Delta(\Delta t - \Delta c)$ typically constant for quadratics), mainly to depict quantitatively and contextually the shortfall between the product and summative views. *Construction of Schemes* - Repeated iterations of the cyclic activities: problematics→actions→reflection whilst coordinating her dialectics between *interpretations*, *multiple representations* and *context* and simultaneously interacting with the researcher fostered the consolidation of Mary's linear schemes, enhanced the consistency and convergence of her multiple representations, engineered the development of tentative quadratic schemes, and extended her reflectively abstracting experiences beyond constant additive rates to include varying ones. Finally, the realistic context invited alternative interpretations and engendered an enriched view of quadratic covariations. Their construction was meaningfully mediated by reflectively abstracting patterns from numeric analyses of tabular forms of rates and facilitated by using appropriate software. These findings provide empirical evidence of how students' developing understanding and construction of quadratic modelling functions (algebraic representations) of situations are initially laden with competing interpretations and conflicting representations but with appropriate technological and teacher support could be scaffolded and guided towards a more convergent and cohesive conceptualisation. This nurturing approach and incorporation of contextual problems into regular classroom activities in schools should be encouraged to foster enriched, conceptual development of key ideas of functions such as rates, to enhance multiliteracies with multiple representations, and to promote reflective and mathematical thinking and discussion in classrooms.

References

Afamasaga-Fuata'i, K. (1991). Students' strategies for solving contextual problems on quadratic functions. Unpublished doctoral dissertation. NY: United States University, Ithaca.

Barnes, M. (2000). Magical moment moments in mathematics: Insight into the process of coming to know. *For the Learning of Mathematics, 20*(1), 33-43.

Behr, M., Lesh, R., Post., T., & Silver, E. (1983). Rational-number concepts. In R. Lesh & M. Landau (eds). *Acquisition of mathematical concepts and processes* (pp. 91-126). Academic Press, Inc.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9-13.

Confrey, J. (1991a). The concept of exponential functions: A student's perspective. In L. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 124-159). New York: Springer-Verlag.

Confrey, J. (1991b). Learning to listen: a student's understanding of the powers of ten. In E. von Glaserfeld (Ed.), *Radical constructivism in mathematics* (pp. 111-138). Dordrecht, The Netherlands, Kluwer.

Confrey, J. (1991c). *Function Probe*: [Computer Program]. Santa Barbara, CA: Intellimation Library for the Macintosh.

Confrey, J. (1994). Splitting, similarity and rate of change: a new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning* (pp. 293-330). NY: SUNY.

Confrey, J., & Doerr, H. M. (1996). Changing the curriculum to improve student understandings of function. In In D. F. Treagust, R. Duit & B. F. Fraser (Eds.), *Improving teaching and learning in science and mathematics* (pp. 162-171). NY: Teachers College Press.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Duit, R., & Confrey, J. (1996). Reorganizing the curriculum and teaching to improve learning in science and mathematics. In In D. F. Treagust, R. Duit & B. F. Fraser (Eds.), *Improving teaching and learning in science and mathematics* (pp. 79-93). NY: Teachers College Press.

Duit, R, Treagust, D., & Mansfield, H. (1996). Investigating students' understanding as a prerequisite to improving teaching and learning in science and mathematics. In D. Treagust, R. Duit, & B. Fraser (Eds.), *Improving teaching and learning in science and mathematics* (pp. 17-31). NY: Teachers College Press.

Hershkowitz, R., Schwartz, B., & Dreyfuss, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*(2), 195-222.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal of Research in Mathematics Education*, *31*(4), 500-508.

Lithner, J. (2000). Mathematical reasoning in task solving. *Educational Studies in Mathematics, 41*,165-190.

National Council of Teachers of Mathematics (NCTM). (1989). *Principles and Standards for School Mathematics*. Reston, VA.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Retrieved 30/1/2005 from WWW: http://my.nctm.org/standards/document/chapter2/index.htm.

New South Wales (NSW). (2002). K-12 Mathematics Syllabus. Sydney: NSW Board of Studies.

Romberg, T. A., Fennema, E., & Carpenter, T. P. (Eds.). (1993). *Integrating research on the graphical representations of functions*. Hillsdale, NJ: Elrlbaum.

Smith, E, & Confrey, J. (1994). Multiplicative structures and the development of logarithms: What was lost by the invention of function? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning* (pp. 333-360). Albany, NY: SUNY.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning* (pp. 3-39). State University of New York Press, Albany, New York.

Steffe, L. P., & D'Ambrosio, B. S. (1996). Using teaching experiments to enhance understanding of students' mathematics. In D. F. Treagust, R. Duit & B. F. Fraser (Eds.), *Improving teaching and learning in science and mathematics* (pp. 65-76). Teachers College Press, Columbia University, New York.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning* (pp.181-234). Albany, NY: SUNY.

Zevenbergen, R., Dole, S., & Wright, R. (2004). *Teaching mathematics in primary schools*. Allen & Unwin.