

Assessment in a Problem Solving Curriculum

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In this paper we elaborate on the ways for assessing problem solving that goes beyond the usual focus on the products of the problem solving process. We designed a 'practical' worksheet to guide the students through the problem solving process. The worksheet focuses the solver's attention on the key stages in problem solving. To assess the students' problem solving throughout the process, we developed a scoring rubric based on Polya's model (1954) and Schoenfeld's framework (1985). Student response to the practical worksheet is discussed.

There has been much interest in mathematical problem solving since the publication of Polya's book about solving mathematics problems (Polya, 1954). The 1980s saw a world-wide push for problem solving to be the central focus of the school mathematics curriculum. However, attempts to teach problem solving typically emphasised the learning of heuristics and not the kind of mathematical thinking used by mathematicians. At the upper secondary and junior college levels, students concentrated on solving national exam-type mathematics problems. The heuristics and problem solving strategies learned at lower levels tended to be ignored instead of being applied in their mathematical engagements, possibly because of the routine nature of the exam-type problems. Therein lays the lack of success of any attempt to teach problem solving within the curriculum. For example, Holton, Anderson and Thomas (1997) proposed a plan for teaching problem solving in New Zealand schools. The New Zealand Ministry of Education developed a national numeracy project which emphasised a problem solving approach and that has now been introduced to the majority of primary schools in the country. However, success is so far limited to the primary level (Ministry of Education New Zealand, 2006) as high stakes examinations have blunted the problem solving approach in mathematics classes at the secondary level.

First of all I think that you have to separate primary from secondary schools. There is a sense in which most primary schools are now using a problem solving approach and are being successful ... Moving further into the secondary school, there are certainly some good teachers who use problem solving especially to introduce new topics but many teachers at that level feel intimidated by the exam system (we have national exams in each of the last three years of school) and so teach at that level in a more 'didactic' manner (Holton, personal communication, 7 December 2006).

Our efforts to meet the challenge of teaching mathematical problem solving to students call for a curriculum that emphasise the process (while not neglecting the product) of problem solving and an assessment strategy to match it, to drive teaching and learning. In this paper, we highlight the 'Practical Worksheet' and present the scoring rubric. (Paper 1 of this symposium series discusses the curriculum design.)

The Practical Worksheet

Tay, Quek, Toh, Dong and Ho (2007), inspired by the idea of a science practical, introduced a modified version, into the mathematical problem solving lesson to focus attention on the process of solving problems. In the implementation, they developed a 'practical' worksheet. The students were encouraged to treat the problem solving class as a mathematics 'practical' lesson.

The worksheet contains sections explicitly guiding the students to use Polya's stages and problem solving heuristics to solve a mathematics problem. A condensed format of the practical worksheet, with all the guiding instructions, is shown below. In the actual worksheet, each section takes up a page, and students may use more of each of sections I, II and III.

I	Understand the problem (UP) (You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.) (a) Write down your feelings about the problem. Does it bore you? scare you? challenge you? (b) Write down the parts you do not understand or that you misunderstood. (c) Write down the heuristics you used to understand the problem.
II	Devise a plan (DP) (You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.) (a) Write down the key concepts that might be involved in solving the question. (b) Do you think you have the required resources to implement the plan? (c) Write out each plan concisely and clearly.
III	Carry out the plan (CP) (You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.) (i) Write down in the <i>Control</i> column, the key points where you make a decision or observation, for example, go back to check, try something else, look for resources, or totally abandon the plan. (ii) Write out each implementation in detail under the <i>Detailed Mathematical Steps</i> column.
IV	Check and Extend (C/E) (a) Write down how you checked your solution. (b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of. (c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Figure 1. Instructions on a Practical Worksheet.

The Scoring Rubric

The scoring rubric focuses on the problem-solving processes highlighted in the Practical Worksheet. There are four main components to the rubric, each of which would draw the students' (and teachers') attention to the crucial aspects of as authentic as possible an attempt to solve a mathematical problem:

- Applying Polya's 4-phase approach to solving mathematics problems
- Making use of heuristics
- Exhibiting 'control' during problem solving
- Checking and extending the problem solved

In establishing the criteria for each of these facets of problem solving, we ask the question, 'What must students do or show to suggest that they have used Polya's approach to solve the given mathematics problems, that they have made use of heuristics, that they have

exhibited ‘control’ over the problem-solving process, and that they have checked the solution and extended the problem solved (learnt from it)?

The rubric is outlined below.

- *Polya’s Stages* [0-7 marks] – this criterion looks for evidence of the use of cycles of Polya’s stages (Understand Problem, etc.), and correct solutions.
- *Heuristics* [0-7 marks] – this criterion looks for evidence of the application of heuristics to understand the problem, and devise/carry out plans.
- *Checking and Extending* [0-6 marks] – this criterion is further divided into three sub-criteria:
 - Evidence of checking of correctness of solution [1 mark]
 - Providing for alternative solutions [2 marks]
 - Extending and generalizing the problem [3 marks] – full marks for this is awarded for one who is able to provide (a) two or more generalizations of the given problem with solutions or suggestions to solution, or (b) one significant extension with comments on its solvability.

The rubric is designed to encourage students to go through the Polya stages when they are faced with a problem, and to use heuristics to explore the problem and devise a plan. They would return to one of the first three phases (see Practical Worksheet) upon failure to realize a plan of solution. Students who ‘show’ control (Schoenfeld’s framework) over the problem solving process earn marks. For example, a student who did not manage to obtain a correct solution would be able to score up to five marks each for *Polya’s Stages* and for *Heuristics*, making a total of ten, if they are show evidence of cycling through the stages, use of heuristics and exercise of control.

The rubric allows the students to earn as many as 70% of the total 20 marks for a correct solution. However, this falls short of obtaining a distinction (75%) for the problem. The rest would come from the marks in *Checking and Extending*. Our intention is to push students to check and extend the problem, an area of instruction in problem solving that has been largely unsuccessful (Silver, Ghouseini, Gosen, Charalambous, and Strawhun, 2005).

Discussion

The Practical Worksheet was introduced to the students in the fifth (of ten) lessons. Both teachers and researchers use the scoring rubric to mark the students’ responses to the problem-of-the-day or problem for homework. We note that most of the students were quick to respond to the emphasis underlined by the scoring rubric. Their responses showed evidence of attempts to go through Polya’s stages (UP – CP – DP cycle), use heuristics, and check and extend the problem. As the weeks passed, we were encouraged by the rising number of students who would take to the responding in such manner. Figure 2 shows a student’s response in *Checking and Extending*. Similar responses were obtained from a quarter of the students (a class of twenty) after the second round of use of the Practical Worksheet. In this example, the problem given to the students is:

Let x and y be positive real numbers such that $x + y = 1$. Prove that $\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9$.

On the other hand, there were a handful of students who were not keen to use the practical worksheet to the extent we had expected. Our interviews with them in the next phase of study might tell us why they took or did not take to the practical worksheet.

What challenges did we encounter in using the scoring rubrics? An inter-rater reliability check based on our ratings and those of the teachers revealed differences in interpretation of responses in *Polya’s Stages* and in *Heuristics* for a handful of students. What does it mean to have completed a cycle of UP-DP? Some students are apt to be ‘untidy’ in the manner in which they write out their attempts. Also, it can be quite a task for students (or anyone for

that matter) to capture their thoughts on paper. As such, it was difficult for the markers to detect the different iterations in the use of Polya's cycles and in the number of heuristics invoked to make progress in problem solving. The matter was largely resolved as the markers understood the rubric better (as is the case in the use of any mark scheme).

The rubric encourages students to formulate significant or hard problems, which they might or might not be able to solve, by awarding marks for them. Thus, effective use of the scoring rubric demands that the marker (or teacher) be sufficiently proficient in mathematics in order to assess the alternative solutions, and the extensions, adaptations, generalizations and ways to solve them. These can be varied especially where the mathematics problem given to the students to solve is rich (see Paper 3 of this symposium series).

IV Check and Extend (a) Write down how you checked your solution. (b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of. (c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them. Check: (a) Used an alternative solution. Alternative solution: $\begin{aligned} \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) &= 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy} \\ &= 1 + \frac{x+y}{xy} + \frac{1}{xy} \\ &= 1 + \frac{1}{xy} + \frac{1}{xy} \\ &= 1 + \frac{2}{xy} \end{aligned}$ $x(1-x) = x-x^2$ $= -(x^2-x)$ $= -\left[\left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]$ $= -\left(x-\frac{1}{2}\right)^2 + \frac{1}{4} \leq \frac{1}{4}$ $\Rightarrow \frac{1}{x(1-x)} \geq 4$ $\Rightarrow \frac{2}{x(1-x)} \geq 8$ $\Rightarrow 1 + \frac{2}{x(1-x)} \geq 9$ $\therefore \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9$ ✓ (Extend See attached foolscap)	Extend: If there are 3 positive numbers x, y, z such that $x+y+z \leq 3$, prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$. Consider the quadratic in t, $(tx+y)^2 + (tx+y)^2 + (tx+y)^2$ $= t^2(x^2+y^2+z^2) + 2t(xy+yz+xz) + (y^2+z^2+x^2)$ The equation above is the sum of 3 squares so it cannot be negative. If the quadratic in t above has 2 distinct roots, then both roots would be negative. (as explained previously in Attempt 2) But this cannot be the case as it is a sum of squares. So the quadratic formula must give either 1 root or no real roots which means $b^2 - 4ac \leq 0$ (discriminant ≤ 0). $4(xy+yz+xz)^2 - 4(x^2+y^2+z^2)(y^2+z^2+x^2) \leq 0$ $\Rightarrow (xy+yz+xz)^2 \leq (x^2+y^2+z^2)(y^2+z^2+x^2)$ Let $x_1 = \sqrt{x}, y_1 = \sqrt{y}, x_2 = \sqrt{y}, y_2 = \sqrt{z}, x_3 = \sqrt{z}, y_3 = \sqrt{x}$ $(1+1+1)^2 \leq (x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ $\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{x+y+z}$ $\because x+y+z \leq 3, \frac{9}{x+y+z} \geq 3$ $\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$ ✓
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Figure 2. Sample of students' work in extending a problem.

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