Developing Conceptual Place Value: Instructional Design for Intensive Intervention

David Ellemor-Collins
Southern Cross University
<david.collins@scu.edu.au>

Robert (Bob) Wright
Southern Cross University
<bob.wright@scu.edu.au>

This paper reports a design experiment focusing on instruction to support low-attaining 3rd- and 4th-graders' development of conceptual place value (CPV). CPV is advanced as an instructional domain more suited than conventional instruction in place value, to support learning of multi-digit mental calculation. Distinctive features of CPV are described; a rich description of a realised instruction sequence for CPV drawn from a ten-week, one-on-one instructional cycle of twenty-nine 25-minute lessons is provided; and modifications to the sequence are proposed.

A significant proportion of primary students are low-attaining in number learning, and there are calls to develop intervention programs to support low-attainers' learning (*Mapping the territory*, 2000; *National Numeracy Review Report*, 2008). Responding to these calls, the Numeracy Intervention Research Project (NIRP) is a design research project with the goal of developing pedagogical tools for intervention in the number learning of low-attaining third- and fourth-graders (8- to 10-year-olds). Within the NIRP we have developed an experimental learning framework for whole number knowledge comprising five key domains: number words and numerals, structuring numbers 1 to 20, conceptual place value, addition and subtraction 1 to 100, and early multiplication and division (Wright, Ellemor-Collins, & Lewis, 2007). For each of these domains we are developing a set of instructional procedures. The analysis reported here is part of the research program developing instruction for the key domain of conceptual place value. The main purpose of the paper is to develop a richly described instructional sequence in this domain.

Background

Mental strategies for multidigit addition and subtraction. Mental computation is an important foundation for learning multidigit addition and subtraction, as it is for all early number learning. Research into mental computation has identified a range of skillful mental strategies that children develop for multidigit addition and subtraction tasks (e.g. Beishuizen & Anghileri, 1998; Fuson et al., 1997). Strategies can be classified into types such as: *jump* (47+18: 47+10→57+3→60+5→**65**), *split* (47+18: 40+10=50, 7+8=15, 50+15=**65**), and *compensation* (47+18: 47+20→67-2→**65**). Using such mental strategies involves a broad knowledge of number relationships (Threlfall, 2002). In particular, each of these strategies depends on ways of relating or structuring multidigit numbers. In the examples above, 18 is partitioned into 10 and 8, 47 is similarly partitioned into 40 and 7, 18 is structured as +20-2, 47 and 10 becomes 57, 47 and 20 becomes 67, and 50 and 15 becomes 65. We describe this kind of activity as *structuring* numbers (Ellemor-Collins & Wright, in press; Treffers & Buys, 2001), to indicate skilled organizing of numbers using relationships, patterns, and other structures.

Structuring multidigit numbers. As in the examples above, multidigit numbers are primarily structured around ones, tens, and hundreds, and the related regularities of decades and hundreds in the number word sequence. Researchers describe different levels of sophistication in students' structuring using ones, tens, and hundreds. In a synthesis

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from four research projects, Fuson et al. (1997) proposed a developmental sequence of students' two-digit conceptual structures. The structures incorporate students' relations between written numerals, number words, and quantities: *unitary* (53 as one, two,…fifty-three); *decade and ones* (one, two…fifty; and fifty-one, fifty-two, fifty-three); *sequence-tens and ones* (ten, twenty, …fifty; and fifty-one, fifty-two, fifty-three); *separate-tens and ones* (five tens and three ones); and *integrated-sequence-separate*. A sixth, incorrect conceptual structure was labelled *concatenated single digit* (53 as five and three). Developing the work of Steffe and colleagues, Cobb and Wheatley (1988) distinguish three levels in children's constructions of ten as a unit, evident in children's thinking in additive tasks. Children operating at level 1 manipulate units of ten and units of one separately, and cannot coordinate them. At level 2, children can coordinate counts or collections of tens and of ones, in the context of representations of the quantities, but they cannot "simultaneously construct a numerical whole and the units of ten and one that compose it" (p.7). Children at level 3 can anticipate, without quantity representations, that a numerical whole consists of tens and ones, and coordinate operations with these units. These models of conceptual structures in students' thinking inform our analyses of learning in this study.

Low-attainers' approaches to multidigit addition and subtraction. Some students do not have success with multidigit addition and subtraction (Cobb & Wheatley, 1988; Ellemor-Collins & Wright, 2007; Fuson et al., 1997; Menne, 2001; Thompson & Bramald, 2002). They may not be able to increment by tens off the decuple. Their additive strategies may involve extensive counting-by-ones, and they may separate tens and ones but be unable to coordinate these units, especially in the absence of base-ten materials. That is, they are limited to level 1 constructions of ten as a unit, and have limited knowledge of structuring multidigit numbers. Thus, in developing intervention in number learning, it is critical to design instruction that supports students' structuring of multidigit numbers (Ellemor-Collins & Wright, 2007). The research reported in this paper is part of a process to design such instruction, with an instructional sequence we call *conceptual place value*.

Conceptual place value. The curriculum topic commonly referred to as *place value* is intended to prepare students for the formal written algorithms for addition and subtraction. For many students, especially low-attainers, place value approaches do not support the development of mental strategies. The formal manipulations of written symbols and materials involved in place value tasks are not yet meaningful for them; instead, the students need to build on their informal number reasoning, which is based in number word sequences and quantities (Beishuizen & Anghileri, 1998; Cobb & Wheatley, 1988; Thompson & Bramald, 2002). We propose an instructional sequence called *conceptual place value* (CPV), to develop students' mental strategies and structuring of multidigit numbers. We draw on the developmental research of Realistic Mathematics Education (RME), which has designed instruction for mental computation with attention to structuring numbers to 100 (Gravemeijer & Stephan, 2002; Menne, 2001; Treffers & Buys, 2001). The CPV sequence focuses particularly on flexibly incrementing and decrementing by 1s, 10s, and 100s. One aim is for students to construct more sophisticated units of ten and hundred, and to be able to coordinate these units. At the same time, an aim is for students to structure a network of relationships between numbers up to 100, and extend this to 1000 and beyond, such that, borrowing Greeno's metaphor (Greeno, 1991), they come to act with multidigit numbers as though knowing their way around a familiar environment.

Instructional settings. We are interested in the design of settings as a context for posing tasks, for students' reasoning, and for didactical discussion. Settings used for instruction in

CPV and multidigit addition and subtraction are described. *Bundling sticks* are wooden craft sticks, tied in bundles of ten. *Base-ten dot materials* are laminated cards with printed dots, including short strips with 1 to 9 dots, 10-dot strips, and 100-dot squares. *Arrow cards* (attributed to Montessori and to Gattegno) are a set of cards with numerals, including 1-9, 10-90, 100-900, and 1000-9000; the cards can be stacked to form 3- and 4-digit numerals. The *empty number line* (ENL) (Treffers & Buys, 2001) is an unstructured line used for marking jump strategies. A *numeral track* is a long strip of paper showing a sequence of numerals, used for reading or checking number sequences.

Instructional design. Our approach with the instructional settings above is informed by the *emergent modelling* heuristic of RME (Gravemeijer & Stephan, 2002). We seek to design instruction in which students can develop informal, context-bound knowledge, and then reflect on their activity, and generalise toward more formal reasoning about numbers. The student's activity of formalising, generalising, or structuring their thinking can be described as *progressive mathematisation*. *Symbolisation* is fundamental to mathematisation (Cobb, 2002; Sfard, 2000). In the development of CPV, numbers can be symbolised with materials, with numerals, and with spoken number words. Symbols do not carry inherent meaning, rather symbols and mathematical meaning co-evolve through the course of a student's problem-solving activity. A significant aspect of instructional design is clarifying the teacher's role in cultivating mathematisation and symbolisation: how to adjust tasks to be usefully problematic for students; how to direct students' attention away from procedural thinking, toward structuring numbers; when to introduce language and inscriptions to support more sophisticated symbolisation.

The purposes of the current study are: (a) to identify key episodes in a student's development of CPV, (b) to describe instruction in CPV that cultivates mathematisation, and (c) to elaborate a hypothetical sequence for CPV instruction.

Method

Design research. The NIRP adopted a method based on design research (Cobb, 2003; Wright et al., 2007), with three one-year design cycles. In each year, teachers and researchers implemented and further refined an experimental intervention program with students identified as low-attaining in their schools. Subsequent analysis of the students' learning is informed by a teaching experiment methodology (Steffe & Thompson, 2000). Analysis of the learning in turn informs our understanding of the instructional settings and tasks, and the development of detailed instructional sequences (Cobb, 2003).

Experimental intervention program. The program year involved (a) in term 2, a range of initial assessments of the students; (b) in term 3, a ten week instructional cycle; and (c) in term 4, final assessments. The major assessment instrument was an individual task-based interview, with tasks addressing the five key domains of the learning framework (Wright, Martland, & Stafford, 2006). The instructional cycle consisted of approximately thirty 25-minute lessons across ten weeks. Each lesson typically addressed three or four domains of the learning framework. In each school, two students were taught individually, six in trios. Individual lessons and assessment interviews were videotaped, providing an extensive empirical base for analysis. In total, the project has involved professional development of 25 teachers, interview assessments of 300 low-attaining students, and intervention with 200 of those students.

Instructional case. For this paper we have selected a single student's teaching cycle as a situation in which to consider issues of instructional design in CPV, for three reasons: (a) to observe instructional episodes in detail, (b) to identify what refinements in the settings and tasks may have contributed to the student's learning, and (c) to develop an account of an extended instructional sequence. The instructional case selected involves Mrs. Parkin and her student Robyn, who was nine years old and in the 4th grade. Theirs is a rich case because Mrs. Parkin contributed significantly to refining the instructional sequence, and Robyn made strong progress in knowledge of CPV and multidigit addition and subtraction over the course of the intervention.

Assessment and Teaching Cycle: Mrs. Parkin and Robyn

Robyn was taught individually for 29 sessions across 10 weeks from July to October. Her initial and final assessments occurred in May and October respectively.

Comparison of Robyn's Initial and Final Assessments

In her initial assessment Robyn identified and could write 3-digit numerals, but not 4-digit numerals. She was not facile at incrementing and decrementing by tens in the range to 1000 and unsuccessful at incrementing by tens beyond 1000 and incrementing by hundreds. With a setting of base 10 dot materials, she had some success with additive tasks, but did not solve a missing addend task (73+x = 100). She did not correctly coordinate units of tens and ones when attempting to solve three bare number tasks (43+21, 86-24, 37+19). In her final assessment, responding to the same tasks in the range 1-1000, Robyn was largely fluent and successful.

Weeks 1 and 2: Two-digit addition and subtraction using bundling sticks

These tasks involved (a) displaying a collection of tens and ones; (b) screening the collection; (c) adding or removing tens and ones; and (d) asking how many sticks resulted. Robyn consistently succeeded with adding or removing tens alone, or several tens and one, or one ten and several ones, but typically was unsuccessful with increments of several tens and several ones. She appeared to calculate the number of tens and the number of ones separately using counting by ones, and to have difficulty coordinating the separate counts.

Weeks 3 and 4: Two-digit addition without regrouping using the empty number line

Mrs. Parkin inducted Robyn into the use of an ENL to record jump strategies for two-digit addition without regrouping but Robyn did not learn to use the ENL autonomously during these lessons. On the task of 23+14 for example, she added 1 rather than 10 and then added 4. On the task of 36+21, she added 20 to 36 and then added 10 rather than 1. She typically had difficulty with adding tens off the decuple and appeared not to understand how the ENL leads to an answer.

Weeks 4, 5, and 6: Base-ten dot materials and arrow-cards

Mrs. Parkin and a researcher inducted Robyn into CPV tasks using base-ten dot materials and arrow-cards. A typical task began with building a 2- or 3-digit number with dot materials and then repeatedly adding or removing a 100-dot square or a 10-dot strip. Robyn's task was to name the number of dots shown after each change, and sometimes to make the number using arrow cards. We describe three lesson episodes in detail.

Lesson 14, Week 5—Base-ten dot materials and arrow-cards

Following several increments Robyn correctly named eight 100-dot squares and a 7-dot strip (807). Mrs Parkin asked Robyn to show the number with arrow cards. The number 307 was currently shown with arrow cards and Robyn replaced the 300-card with the 800-card. Mrs Parkin placed out another 100-dot square, after which Robyn said “nine hundred and seven”. Mrs Parkin again placed out another 100-dot square. After six seconds Robyn said “Umm…nine…. One thousand and seven.” On request from Mrs Parkin Robyn then made 1007 with arrow cards:

Mrs. Parkin: Why is there [points toward the arrow cards for 1007] all those zeroes?

Robyn: Because there’s a thousand, and the hundred and ten have not been…because there’s no numbers that are a hundred and…cos there’s…cos there…umm…because there’s…umm…because there’s no like one thousand two hundred and twenty-seven, it just has, it’s just one thousand and seven.

Mrs. Parkin: Good. There’s thousands, but there’s no [points to the 0 in the hundreds place, as shown in Figure 1a]…

Robyn: Hundreds.

Mrs. Parkin: And there’s no [points to the 0 in the tens place]…

Robyn: Tens.

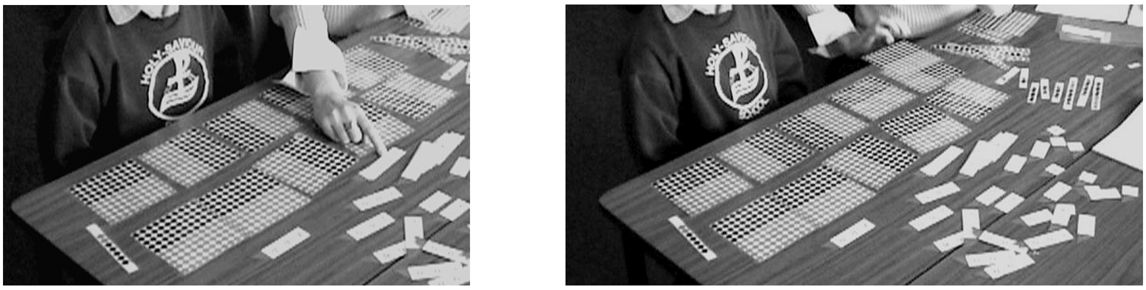

Figure 1. Lesson 14: 1007. (a) “There’s thousands, but there’s no…” (b) “If I added in another one [100]?”

Mrs Parkin then placed out another 100-dot square (1107) as shown in Figure 1b. Robyn looked at the dot materials and said “Two thousand and seven”, looked up, said “umm”, looked down at the dot materials again, and said “…one thousand two hun…one…t…. One thousand, one hundred and seven.” Mrs. Parkin then asked Robyn to make the number (1107) with the arrow-cards which she did without error. Mrs Parkin then added a 100-square three times. As before, Robyn succeeded with difficulty in naming the number of dots after each 100-square was added. On being asked to make the number with arrow cards, Robyn first made 4407, but after reading the numeral corrected the arrow cards to 1407.

Lesson 16, Week 6—Base-ten dot materials and arrow-cards

Following several increments there were two rows of five 100-dot squares, two more squares, seven 10-dot strips, a 3-dot strip and a 2-dot strip. Robyn named the number correctly (1275), and made it with arrow cards.

Mrs. Parkin: You’ve got one thousand two hundred and seventy-five? Now, let’s count backwards. [Removes a 100-dot square.] Now what?

Robyn:	One thousand one hundred and seventy-five.
Mrs. Parkin:	[Removes another 100-dot square, leaving the two rows of five squares.]
Robyn:	Nine hundred and seventy-five.
Mrs. Parkin:	Check that.
Robyn:	[Shakes a finger five times over the 100-dot squares, in coordination with saying subvocally—] Two, four, six, eight [and then saying aloud—] Ten hundred…one thousand…one thousand one hundred…. One thousand and seventy-five.
Mrs. Parkin:	And what would that look like in the arrow-cards?
Robyn:	[Removes the 200-card from the arrangement for 1275, making 1075.]

Mrs Parkin made the following succession of 11 decrements and Robyn successfully named the number after each decrement: 1075 to 975, to 965, to 865, to 855 to 845, to 842, to 832, to 822, to 812, to 802, to 502.

Mrs. Parkin:	What if I took away a ten? [On one of the five 100-dot squares, she covers one column of ten dots with a blank card.]
Robyn:	Umm, five hundred…[Pauses for 7 seconds, looking back along the row of 100-dot squares]…Four hundred and ninety-two.

Lesson 18, Week 6—Arrow-cards

Mrs. Parkin asked Robyn to make the numeral 6042 on the arrow-cards, and then to make the number 100 less. Robyn made 5042 and then indicated that she was incorrect, whereupon she was directed to use the dot materials. Robyn then solved three tasks involving the arrow-cards alone: 100 less than each of 2034, 8056 and 1025.

Weeks 7 and 8: Numeral track and ENL

At the end of Week 6, Mrs. Parkin introduced the numeral track for number word sequence and numeral sequence tasks by 2s, 5s, and 10s, working over the hurdles at 1000 and 1100, and continued with this in Weeks 7 and 8. As well, Mrs. Parkin posed 2-digit addition and subtraction tasks requiring regrouping. Robyn solved the tasks successfully using jump strategies on the ENL, initially with bundling sticks, and later without.

Discussion

Robyn's progress in structuring multidigit numbers. Robyn made remarkable progress in her intervention, as evident in the comparison of her initial and final assessments. During the teaching cycle Robyn learned to increment and decrement by 1s, 10s and 100s in the range 1 to 1000 and beyond, and to use jump strategies to solve bare number 2-digit addition tasks. She developed a Level 3 unit of 10 (Cobb & Wheatley, 1988), and sequence-tens and separate-tens conceptions (Fuson et al., 1997). Having documented the instructional sequence as it was realised by Mrs. Parkin, we can analyse what made the instruction effective, and elaborate our instructional design for CPV.

Linking number word, numeral, and quantity. Number words, numerals, and material quantities, are three primary ways we symbolise numbers. Establishing consistent links between these three symbolisations is recognised as a foundation of number knowledge (e.g. Fuson et al., 1997). The CPV tasks created opportunities for Robyn to make these links. Every increment or decrement of the dot materials involved a relation between the quantity and the number word. For example, in lesson 14, at the increment of a 100-dot square in the material, from 907 to 1007, Robyn had to start saying "one thousand" but stop saying the word "hundred", while at each of the four subsequent increments, to 1107, 1207, 1307, and 1407, Robyn had to keep re-organising her number words so that the thousands did not change, while the hundreds did. In lesson 16, to check 100 less than 1175, Robyn counted the dot-materials, so that "two, four, six, eight, ten" 100-dot squares became the number word "one thousand". Activity with the arrow cards made the links to numerals more explicit, as in lesson 14, Mrs. Parkin's questioning about the zeroes in the numeral 1007, and Robyn's reading and correcting her numeral 4407. We think this work in developing different symbolisations strengthened Robyn's structuring of multidigit numbers.

Distinctive features of CPV. Conventional place value tasks typically involve talking about the construction of numerals, and making links between numerals and base-ten materials. What is distinctive in CPV tasks is the attention to number words, and to incrementing and decrementing. The incrementing and decrementing demands the coordination of units. For example, at the end of the lesson 16 episode, the persistent variation of the unit of decrement required a coordination of the hundreds, tens, and ones, and the final decrement of 10 from 502 to 492 required the hundreds change at the same time as the tens. We think knowledge of the dynamic relations between 1s, 10s, and 100s in multidigit numbers is a good basis for mental calculation.

Developing an instructional sequence in CPV. The instructional sequence could begin by incrementing/decrementing just 10 or 1 in the range 1-100, using the combined base-ten dot material and arrow-card settings. The CPV tasks can then be developed in three directions. (a) The range of numbers can be extended, to 1000, then beyond 1000, as Mrs. Parkin did so productively with Robyn. (b) A second development of the tasks is to formalise the setting with a shift toward bare written numerals. Mrs. Parkin progressively reduced the use of the base-ten dot materials and bundling sticks over weeks 6, 7, and 8, moving to work with numerals or the ENL alone. (c) A third development of the tasks is to increase the complexity of the increments/decrements to multiple tens or hundreds, and to combinations of ones, tens, and hundreds. During week 6, Mrs. Parkin moved from increments of a single ten or hundred immediately to 2-digit additive tasks. We hypothesise that a more graduated progression would have been more supportive. This elaborated instructional sequence has been trialled successfully in the most recent teaching cycles in the project.

This study provides a rich description of instruction that resulted in significant learning in the domain of conceptual place value. We conclude that the proposed instructional sequence for CPV is viable as a basis for intensive intervention and we will continue to refine it.

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