Teaching the Distributive Law: Is Fruit Salad Still on the Menu?

Helen Chick
University of Melbourne
<h.chick@unimelb.edu.au>

Introducing algebra in high school is a teaching challenge, where appropriate explanations and representations are essential. This study examines teachers' reactions to a page of explanation about the distributive law from a textbook, and investigates their pedagogical content knowledge about the teaching of algebra. Teachers were aware that students have difficulties with algebra, but the study shows that some of their strategies for teaching it are mathematically unsound. In particular, "fruit salad" approaches are still prevalent.

The topic of algebra is notorious among teachers for being difficult to teach. The level of abstraction, the ideas of generality, and the use of a new symbolism all contribute to the challenge teachers experience helping students make the step from the concrete number world to the abstract world of real number properties, algebraic expressions, pronumerals, and variables. The models and explanations that teachers use reflect their understanding of how children learn, what concepts are critical to illuminate, and what foundations such a model is likely to build. These concerns lie in the domain of pedagogical content knowledge (Shulman, 1986); as we shall see, this is complex for the topic of algebra.

Background

It is essential that students come to understand that letters stand for numbers in algebra: as generalised numbers, unknown numbers, or variables. That students have difficulty with these ideas has been known for at least 30 years (e.g., Küchemann, 1981). That students have continued to have difficulty in the intervening time is also well known. MacGregor and Stacey (1997), for example, conducted a large-scale study of 11-15 year old students' understanding of algebraic notation. Building on Küchemann's work, they found that students interpreted letters in algebra in a variety of ways: letter ignored, numerical value, abbreviated word, alphabetical value, different letter used for each unknown, unknown quantity, letter as label, letter equals 1, and letter as a general referent. They were concerned that older students who had experienced some years of algebra learning did not know what letters represent in algebra, but also pointed out that students apply quite logical reasoning in their incorrect use of algebra.

One of the best known of the misconceptions is the "letter as object" misconception, described by Küchemann (1981), in which the letter, rather than clearly being a place-holder for a number, is regarded as being an object (the abbreviated word misconception is such an example). The term "fruit salad algebra" is sometimes used for this misconception, infamously presented in examples such as "a is apples and b is bananas, and so $3a + 2b$ is like 3 apples and 2 bananas, and since you can't add apples and bananas we just write it as $3a + 2b$." One difficulty here is that $3a$ in algebra does not represent 3 apples, but three times an unknown *number*. The second difficulty here concerns the mathematical idea of closure: in saying we cannot add apples and bananas we contradict the fact that $3a + 2b$ *is* the sum. The letter as object misconception may be reinforced by formulae like $A = l \times b$, where A = area.

MacGregor and Stacey (1997) found that the teaching materials in one of the schools of their study used a "fruit salad algebra" approach. The performance of students from that

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school reflected a prevalent “letter as object” misconception. MacGregor and Stacey recommended that teaching materials should avoid using such analogies, thus reiterating a call made by MacGregor in 1986. A later report (Stacey & MacGregor, 1999) highlighted three textbooks of the day with “letter as object” explanations for algebra.

Teachers have “a responsibility to ensure that students’ first experiences of using letters in algebra lay the foundation for a coherent structure of algebraic knowledge” (MacGregor & Stacey, 1997, p. 18). Kieran (1992, 2007) has called for more research on teachers’ knowledge regarding algebra and its role in algebra instruction. In 2007 she pointed out that research focused on the algebra learner and the algebra teacher are framed in quite different ways. In reviewing the findings of recent research in the area of pedagogical content knowledge (PCK) Kieran emphasised the importance of teachers having strong content knowledge for algebra but with the caution that in the absence of well developed pedagogical content knowledge this may lead to a tension if “pedagogical decisions [are] based [solely] on the structure of the mathematical domain rather than on an understanding of the actual ways in which students think” (p.741). Of course, if the structure of the mathematical domain is not well-understood the problem is magnified.

This concern may be illustrated by the findings of Menzel and Clarke (1998) who conducted an in-depth study of a small number of mathematics teachers. These teachers typically described algebra in terms of curriculum topics and emphasised procedural knowledge. Very little mention was made of the need for a foundational understanding of the meaning of letter; indeed some of those interviewed did not believe that algebraic activities need involve written symbols at all. The issues and topics to which these teachers drew attention did not suggest a focus on the meaning which students give to letters in algebra. Doerr (2004, p. 267) further highlighted that “despite large body of research that demonstrates the ineffectiveness of teaching algebra as procedures that are disconnected from meaning and purpose, much of school algebra is still taught as disconnected procedures”.

Artigue, Assuade, Grugeon and Lenfant (2001) and Doer (2004) describe a framework highlighting the importance of teachers knowing—among other things—algebra content and structure, the place of algebra in mathematics, the nature of valuable algebra tasks for learners, the development of students’ algebraic thinking, students’ interpretations and misconceptions of algebraic concepts and notation, resources and different instructional representations. This framework contains algebra-specific aspects of PCK, within themes like those organised from the PCK literature, such as “knowledge of explanations”, “knowledge of structure and connections”, and “knowledge of students’ thinking” (see, e.g., Shulman, 1986; and, more recently, Chick, 2007 and associated references).

With this background in mind, this study seeks to examine whether teachers recognise some of the issues associated with learning the “meaning of letters”, what approaches they take when teaching algebra, what PCK for teaching algebra they exhibit, and the extent to which they can identify students’ learning needs in algebra.

Method

As part of a larger questionnaire and interview protocol intended to explore the PCK of secondary mathematics teachers, 35 teachers from 3 schools were presented with a page from a current Australian year 8 mathematics textbook[i]. The textbook presented two examples illustrating the distributive law. The first explanation had a picture of 6 apples beside 4 bananas and then repeated the picture with a horizontal line to show 2 groups, each of 3 apples and 2 bananas. The number of apples and bananas in both cases was written in words and the expressions $6a + 4b$ and $2(3a + 2b)$ were alongside the first and second pictures respectively, followed by the identity $2(3a + 2b) = 6a + 4b$. For the rest of this paper this will be referred to as the "apples and bananas" explanation. The text asserted that this illustrated the distributive law, and that there is a factorised and an expanded form of this law, but did not identify which form is which. It then gave the general expression of the distributive law as $a(b + c) = ab + ac$, with arrows linking a with b and a with c over the expression $a(b + c)$. The second explanation of the distributive law followed this text, and used grid paper to show a 5×7 rectangle as being composed of two smaller rectangles, one 5×4 and the other 5×3. The total number of squares was described as being both $5 \times (4 + 3)$ and $5 \times 4 + 5 \times 3$. This will be referred to as the "grid" explanation.

For this study the teachers were asked to give written responses to three questions:

- Identify any positive aspects of the way the distributive law is presented here.
- Identify any negative aspects of the way the distributive law is presented here.
- Would you use these explanations? Explain.

A follow-up interview, conducted with all but two of the 35 teachers who had completed the whole questionnaire, allowed further discussion of each teacher's responses. These interviews followed a semi-structured protocol. There were three standard questions, listed below.

- What did you think about that page?
- What are some of the "issues" you have noticed with textbooks?
- How do you usually explain the distributive law?

In some cases where teachers had indicated support for the "apples and bananas" explanation, teachers were also asked if they thought there might be any problems with that approach.

Content analysis (Bryman, 2004) was conducted with the written data to identify common themes in response to the three questions on the questionnaire about positive and negative aspects of the page and whether or not teachers would use the explanations. Interview data that also addressed the questionnaire items were included in this content analysis. Simple frequency counts were obtained for each emerging theme. Additional themes associated with pedagogical content knowledge for algebra teaching were identified in the interview data. The approach here was phenomenographic, and comments that particularly illuminated the themes were selected as exemplifying cases (Bryman, 2004).

Results

Teachers' Views About the Approaches

Table 1 shows the positive features of the textbook page identified by the teachers. All but five of the 35 teachers mentioned the visual depictions of the distributive law as a good attribute of the page. One teacher's comment suggested that this valuing of the visual may be due to teachers' recognition that students have different learning styles; she said "Visuals are good … I'm a visual learner myself, so I loved all the visual stuff, and this makes a lot of sense to me". Teachers' beliefs about how to help students make the transition to the abstract nature of algebra may also be reflected in the fact that nearly half of them felt that the visual aspects of the page provided a concrete bridge or link between the concrete/physical and the general algebraic principle.

The importance of having multiple alternative explanations was mentioned by seven of the teachers. Curiously, however, two of the teachers who regarded having a range of alternatives as good, expressed concern about the "apples and bananas" explanation. This may be because the general principle of having alternative explanations over-rode the teachers' view of the problematic nature of one of the explanations, or because they viewed the arrow-highlighted presentation of the algebraic expansion $a(b + c) = ab + ac$ as an explanation as well.

Four teachers felt that the ideas were sequentially presented, which they regarded as a positive feature. This also suggests that they viewed the grid explanation as harder than the "apples and bananas" explanation. A similar number of teachers felt that it was good to that there were objects (e.g., apples) in place of the pronumerals, perhaps also reflecting a belief that this makes the explanation more concrete. This was usually stated without recognising the "letter as object" misconception, although one teacher's response began to address the contradictions in this view:

> [The apples and bananas] is definitely a good way of starting … it uses items replacing the pronumerals […] so it's actually objects … Maybe in the long term if they kept that up without switching over to using the pronumerals it might not [work] when the letters end up representing values.

Table 1
Positive aspects of the textbook page identified by the teachers (n=35)

Identified positive feature	Number (%)
Visual presentation was good	30 (86%)
Connected the ideas to a physical/practical/concrete context	17 (49%)
Alternative explanations were presented	7 (20%)
The grid method connects nicely to other representations/concepts	7 (20%)
The material was sequentially presented	4 (11%)
Pronumerals were recognisable items/objects	5 (14%)

As shown in Table 2, there was more diversity in the teachers' opinions about problematic aspects of the textbook page. A third of the teachers felt that the presentation of the grid explanation was "busy", unclear, or intrinsically more complicated. In particular, about half of this group felt that it required higher order thinking from the students, suggesting that the teachers—correctly or not—were at least mindful of the cognitive demands of the explanation. One teacher commented that "I had to really concentrate when I was reading [the grid explanation] […] [it] is a big jump […] I think to understand [the grid students] already need an understanding of the distributive law". In this particular case, the teacher has failed to recognise that students *should* already have an intuitive understanding of the distributive law based on their understanding of number, and that the algebraic expression of the law describes a general identity applying to all real numbers.

There were 16 teachers who expressed concern about the "apples and bananas" explanation, with three themes evident in their comments. Only nine (about a quarter of the whole group of teachers) clearly expressed recognition of the "letter as object" phenomenon, or stated that the pronumeral needed to be a place-holder for a number. One of these actually expressed surprise that the textbook page had gone to print. An additional six recognised that the "apples and bananas" approach was misleading or not ideal, but could not give a clear justification for this belief. Finally, two teachers (including one also included in the "letter as object" group) felt that the "apples and bananas" explanation was "boring" or "naff". One of these teachers commented

> "I don't really like the fruit thing, generally […] it's almost just a little mathematical idea that someone somewhere once mentioned that somehow turned into a little bit of a meme that goes around mathematics education […] it's very powerful […] I do like the idea that *a* is in the place of something; it is a placeholder rather than a particular object […] It's an overly concrete example for something where I'm starting to want to be a little bit fluid and theoretical and open […] I'm not really a big fan of the fruit equals mathematics notion"

Nine of the teachers expressed concern about the language and/or reading requirements of the textbook page, and the demands this might place on students. One teacher commented that she felt that textbooks generally had "too much in them" and are "busy"; she did not believe that any students actually read them. Seven teachers felt that necessary explanatory material had been omitted from the page, such as discussing the omission of the multiplication sign, and defining "expanded" and "factorised". Two teachers felt that the two explanations—"apples and bananas" and "the grid"—could have been more closely aligned in order to parallel each other. Finally, ten teachers identified a range of other short-comings, such as the limited generality evident in the single example of the grid explanation, and the appearance of a *c* in the general statement of the distributive law (one teacher suggested students would ask "Where did *c* come from? Is that carrots?").

Table 2
Negative aspects of the textbook page identified by the teachers (n=35)

Identified negative feature	Number (%)
Rectangle grid explanation was busy/unclear/more complicated	12 (34%)
Language and/or reading requirements were too hard	9 (26%)
"Letter as object" presentation was a problem	9 (26%)
Necessary extra explanation was identified as missing from the text	7 (20%)
Use of apples and bananas was misleading/not ideal	6 (17%)
Use of apples and bananas was boring/passé	2 (6%)
Connection between the two explanations was unclear	2 (6%)
Other negative feature	10 (29%)

Table 3 indicates whether or not the teachers would use either of the suggested explanations from the textbook page. As is evident, "fruit salad" still does, indeed, appear on the mathematics teaching menu, with three-quarters of the teachers stating that they would use the "apples and bananas" explanation. This group includes two teachers who had successfully identified the "letter as object" issue as being problematic. Some of the teachers indicated that they would use the "apples and bananas" approach with weaker students or as an introduction before moving to numerical examples. One said that it is a good teaching tool because you can highlight that "you can't add apples and bananas; it's gotta stay as just $a+b$"; another said she would reinforce that "a could equal any number";

and a third stated that although she would use it she knows that students "need to switch from objects to pronumerals". Support for the grid explanation was significantly lower.

Table 3
Teachers' support for the use of the explanations (n=35)

Explanation	Willing to use explanation	
	Yes	No
"Apples and bananas" explanation	26 (74%)	6 (17%)
Rectangular grid/array explanation	19 (54%)	10 (29%)

Note. Some teachers did not give a clear indication about whether or not they would use a particular explanation, so the total of the "Yes'" and "No" responses in each row of the table is less than 100%.

Other Indications of Pedagogical Content Knowledge

As described earlier, teachers' interview responses revealed further aspects of their pedagogical content knowledge for teaching algebra. The data show what teachers might be thinking and doing in their algebra teaching practice, and have been grouped within PCK themes.

Knowledge of Alternative Explanations. By far the most common alternative approach to explaining about the distributive law was the use of multiple numerical examples. Teachers commented on the use of "small integers", or the link to mental computation strategies (such as calculating 5×104 by working out 5×100 and 5×4 and adding). A couple of teachers expressed the intention of having students make the generalisation to the distributive law themselves, with one saying he would proceed to prove the result. This particular teacher emphasised proof as being at the heart of mathematics (thus also showing knowledge of structure and connections), although he did not realise that his proposed proof would only work for the special case where a is a positive integer. One teacher mentioned teaching algebra by using envelopes with a number of paperclips within; another mentioned using a "short multiplication" method in which the expanded form is worked out by applying an algebraic analogue of the short multiplication algorithm (one expression on each line; each term is equivalent to a place value position). A few teachers mentioned teaching the procedure, by rote, with the arrows to indicate the expansion. One teacher mentioned using a smartboard, but did not actually discuss the mathematical approach he would take with the explanation.

Knowledge of Structure and Connections. As shown in Table 1, and suggested above, seven of the teachers recognised connections between the grid explanation and other areas of mathematics. These other areas included the mental decomposition strategies and the short multiplication algorithm already mentioned, and the use of a rectangular area model to demonstrate how to expand the product of two linear factors in a quadratic expression.

A lack of knowledge of structure and connections was revealed in response to the question—not asked of all the teachers—concerning whether the "apples and bananas" approach leads to later problems. One teacher admitted that "apples × bananas" did not make sense, but was happy with the idea of "apples × apples". Another spoke enthusiastically about being able to highlight that you can't mix an "alligator-squared" with an "alligator", but did not comment on the incongruity of "alligator-squared". In both these cases the closure issue again lurks in the background A third teacher recognised that "you can't take apple to the power of zero". These perceived shortcomings were not enough, however, to prevent teachers from supporting the approach, with the last teacher indicating that he valued the more concrete nature of the explanation, presumably for his students' benefit.

Two of the teachers expressed support for the "apples and bananas" approach because here *a* and *b* stand for something whereas *x* and *y* do not. As one of these teachers claimed:

> "It gets away from the confusion of what is x and what is y, because x and y, for some reason, don't seem to stand for anything. […] What does that word [pronumeral] actually mean? […] It's standing for something we don't know. What is it? Who knows? It's for a number, it's for a thing, it could be anything".

Finally, one teacher wrote that he felt that expanded and factorised forms are building blocks for the distributive law; his response failed to identify that these are, rather, fundamental components of it.

Knowledge of Students' Thinking. The responses from many of the teachers indicated that they recognised and were concerned about students' difficulties with algebra, particular as a more abstract field of mathematics. This was reflected in the frequency of responses supporting the practical or concrete nature of the explanations on the page (validity notwithstanding), and other expressions of concern about presenting the material in a way that "kids will accept it". In most of the cases, the teachers' preferences for one method over another appeared to be a reflection of their desire to have students able to execute algebraic procedures, regardless of understanding, as if this is "doing" algebra, in some sense.

Conclusions

Before commenting on the implications of this study it is important to recognise that the "fruit salad algebra" approach was present in the textbook. Unfortunately, despite the long-recognised problems with the approach and the calls for its use to end (MacGregor, 1986; Stacey & MacGregor, 1999), it still appears in contemporary texts. If textbooks are perceived as being an authority in mathematics then this gives legitimacy to the "apples and bananas" method. On one occasion during the interviews it became apparent that one of the teachers found the explanation appealing, and the author had to intervene to prevent the teachers' conversion to the infidel fruit salad.

The responses of many of the teachers, however, suggests that "fruit salad algebra" approach is well-entrenched in teaching culture, perhaps sanctioned by some textbooks, but perhaps also perpetuated through teachers' knowledge sharing or because that is how teachers were themselves taught. Its continued use may well occur because the "apples and bananas" analogy is perceived by teachers as being accessible for students, as evidenced by some of their comments. Particularly strong evidence for this comes from the fact that two of the teachers who could articulate the "letter as object" problem—together with a further four teachers who felt that the "apples and bananas" explanation was not ideal—still indicated that they would use the approach. One of the former two teachers spoke of attending a professional development session in which the "letter as object" misconception had been discussed and "fruit salad algebra" strongly discouraged. His continued use suggests that some teachers may be unconvinced about the dangers of "fruit salad algebra", that they may lack confidence with or knowledge of alternative strategies, or that they do not believe that such alternatives are as "helpful" for students.

Although many teachers were able to see that the distributive law reflects a property of the real number system—as shown by their suggestion to use numerical examples to establish the law—this knowledge was not widely evident. This, together with the belief in the "apples and bananas" approach and the limited acceptance of the grid method, may reflect deeper problems associated with knowledge about the structure of algebra. The issue of closure is an intriguing one. Even supposing a fruit salad approach is *not* taken, it is not hard to imagine a teacher saying that you cannot add $3a$ and $4b$, and yet we *can* multiply $3a \times 4b$ to get $12ab$. This is somehow a closed answer in a way that $3a + 4b$ is not, because *something* has been able to be "done". Making sense of multiplication of pronumerals seems to be a critical component of teachers' PCK; the acceptance of "apples × apples" and "alligator-squared" could be investigated further.

The teachers in the study all seemed to want their students to succeed in algebra; most of their choices were justified by referring to perceived advantages for student learning. What remains, as has been the case for 30 years, is to convince them of the disadvantages of "fruit salad algebra" and to help them use strategies that they recognise are mathematically correct and pedagogically productive.

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[i] Because textbooks have strengths and weaknesses, a bibliographic reference to this particular book has not been given, lest assumptions be made about its relative quality in comparison to other texts.