Success and Consistency in the Use of Heuristics to Solve Mathematics Problems

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The ability to solve mathematics problems is the main goal of mathematics education in many countries. This ability depends on coordinating several types of knowledge and mathematical processes, especially heuristics. Commonly used heuristics include guess and check, draw a diagram, logical argument, and simplifying the problem. This paper describes the heuristics used by a sample of Primary 5 (n = 221) and Secondary 1 (n = 64) Singapore pupils to solve problems like this one: "There are 100 buns to be shared by 100 monks. The senior monks get 3 buns each and 3 junior monks share 1 bun. How many senior monks are there?" The pupils solved two sets of problems; the second set consisted of parallel problems to the first set but was administered a few months later. The pupils' written solutions were analysed according to the heuristics used. A comparison of the heuristics used between the two parallel tests shows that some pupils did not use similar heuristics to solve parallel problems. This issue of consistency in heuristic use should be further researched to unravel its implications for the teaching of heuristics.

Background

Most countries include problem solving as one of the key objectives of school mathematics education. But many pupils at all school levels have difficulty solving unfamiliar or so-called "non-routine" problems. Confronted with undergraduate pupils who did not know how to begin to solve a novel problem, Polya (1957) devised a 4-step model that has impacted enormously on the teaching of problem solving in schools over the past half century. A key element of his problem solving model is the use of heuristics. A heuristic is a generic rule, such as Think of a Similar Problem, Draw a Diagram, and Guess and Check, that can be used to solve different types of problems, though there is no guarantee that it will be successful. In some countries such as Singapore, the teaching of heuristics has become an important step towards developing problem solving skills among the pupils. In Singapore (http://www.moe.gov.sg/cpdd/syllabuses.htm), a dozen of heuristics are taught in primary and secondary schools, and the Primary School Leaving Examination (PSLE) taken by all pupils in Singapore at the end of their primary school education contains "challenge problems" that require the application of heuristics. While the effectiveness of the teaching of heuristics is a controversial pedagogical and research issue, this paper only examines how successful a sample primary and secondary pupils were in solving problems using some of these heuristics. In normal classrooms, the teachers should be able to know how successful their pupils are able to apply heuristics to solve problems, but such information is rarely compiled and shared beyond their classroom settings. This paper fills a gap by providing systematic information about heuristic use by pupils.

The Study

The data in this paper were gathered as part of the research project entitled "Developing the Repertoire of Heuristics for Mathematical Problem Solving" (MPS) conducted in Singapore in 2004. This MPS project had three main components.

The first component examined whether the pedagogical practices of mathematics teachers would support mathematics problem solving. The participating teachers from three primary schools and two secondary schools, with two classes per school, were observed in two rounds. Each round of observations covered several lessons that made up one teaching unit (Ho & Hedberg, 2005). The teachers attended a 3-hour workshop in between the two rounds, separated by about three months.

The second component, the concern of this paper, looked at pupils' solutions to the mathematics problems administered at the end of each round of observations. After each test, the pupils answered a questionnaire about their perceptions of the problem solving process and perceived difficulty of the problems. The aim was to relate pupils' heuristic use to their affective and metacognitive responses. These two rounds of tests are referred to as "pre-test" and "post-test" in this paper, although it is not appropriate to consider this design a rigorous experiment due to the short duration of the intervening workshop. Each test consisted of nine problems. The problems in the post-test were parallel but not equivalent to those in the pre-test, with modifications in the story line and numbers used. One of the problems was identical in both tests.

The pupils were given 40 minutes to solve the problems and 20 minutes to answer the questionnaire. They were instructed to show full workings for the problems. The pupils seemed to react quite differently to the tests. In one school, the post-test was administered after an examination, and the pupils were not motivated to complete the test. After the pre-test, a teacher commented that the statement "This is NOT a test" would affect pupils' attitude towards completing the test to their best ability. Hence, this statement was removed from the post-test. These matters about test administration may affect the pupils' motivation to complete the tests and the results may not reflect their true performance. Due to some contingencies, post-test data could not be collected from one of the two secondary schools, resulting in a sample mortality of two classes. The final sample consisted of 221 Primary 5 pupils (11 years old) and 64 Secondary 1 pupils (13 years old) who took both tests. Since no generalisation to the respective pupil population was intended, this loss was not a serious flaw, though certainly less than ideal.

The third component consisted of videotaping pairs of pupils solving mathematics problems. The aim was to capture their metacognitive processes through talk aloud protocol (Lioe, Ho, & Hedberg, 2006).

Analysis of Pupils' Written Solutions

Teachers and researchers often analyse pupils' written solutions to better understand the problem solving strategies used by the pupils. For example, Covi, Ratcliffe, Lubinski, and Warfield (2006) categorised pupils' written solutions to a given problem and found that among the codeable strategies, Guess and Check was most prevalent.

In this study, the pupils' scripts were analysed according to the types of heuristics used. Five major types of heuristics were first determined based on previous experience and selected scripts: Systematic Listing, Guess and Check, Equations, Logical Argument, and Diagrams. Not all the five heuristics were suitable for every given problem.

The research assistant coded all the scripts. When uncertainty arose, the author and the research assistant discussed the solution and decided upon its code to ensure consistency. Correct solutions or completely wrong ones were relatively easy to code. There was some difficulty in coding partially correct solutions. However, this was not a particularly serious issue with mathematics, in contrast to the more problematic situation of coding affective variables and classroom observations that requires stronger subjective interpretations. The next section describes the findings for only one particular pair of problems, where only the first four strategies were appropriate.

Analysis of One Pair of Problems

Pre-test problem (*Monks*): There are 100 buns to be shared by 100 monks. The senior monks get 3 buns each and 3 junior monks share 1 bun. How many senior monks are there? [Answer: 25 senior monks and 75 junior monks]

Post-test problem (*Monkeys*): The zoo keeper gave 80 bananas to 50 monkeys. The big monkeys ate 2 bananas each, and 3 small monkeys shared 2 bananas. How many big monkeys are there? [Answer: 35 big monkeys and 15 small monkeys]

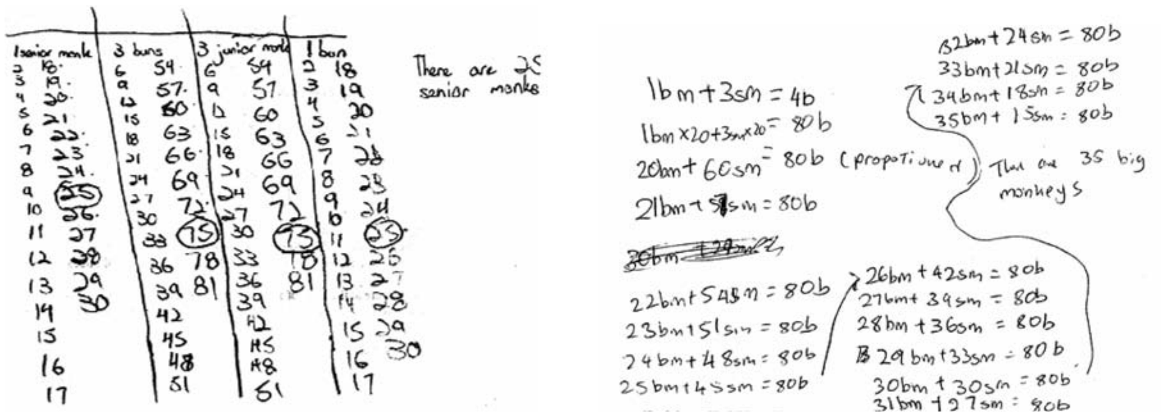
These two problems are similar in structure. They require pupils to coordinate three pieces of information: (a) total number of persons (monks) or animals (monkeys), (b) total number of items (buns/bananas), and (c) ratio of relevant items according to the given conditions. However, the post-test problem is more difficult than the pre-test one because of two factors. First, the numbers of bananas and monkeys are no longer the same, as in the Monks problem. This requires more complex proportional thinking. Second, the Monkeys problem involves a slightly more complicated fraction, $\frac{2}{3}$, compared to $\frac{1}{3}$ in the Monks problem. Nevertheless, the same heuristic can be used to solve either problem.

C1: Systematic Listing (Primary: 2, 2; 1, 8; Secondary: 3,-; 2, 2)

For coding purpose, Systematic Listing must involve at least three consecutive items, to distinguish it from Guess and Check, which may be conducted in an ad hoc fashion including hitting on the correct answer in one lucky or unexplained guess. The figures above show the success rates. For the primary group, two pupils applied this heuristic to get the correct answer for the Monks problem and two got a partially correct solution;

one pupil got the Monkeys problem correctly, while eight got a partially correct solution. For the secondary group, three pupils got a correct solution for the Monks problem and there was no partially correct solution; two pupils got a correct solution and two pupils got a partially correct solution for the Monkeys problem. These values show that this heuristic was used by very few pupils.

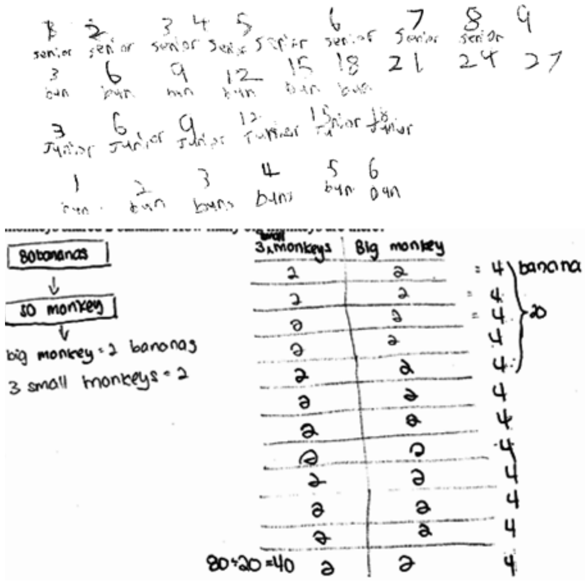
Two correct solutions are shown in Figure 1. In Figure 1(a), the pupil kept the ratio between the number of persons and the number of items constant (1 senior monk : 3 buns; 3 junior monks : 1 bun). In Figure 1(b), the pupil began with the simplest case and varied one of the conditions (number of big monkeys) and adjusted the values accordingly. These two solutions show that Systematic Listing can be carried out in more than one way, depending on which aspect of the problem to work on. Note that the second solution used letters (bm, sm, b) to stand for entities rather than numbers; this is the common *mis*use of algebraic letters to represent objects in the so-called "fruit salad algebra" (MacGregor, 1986).



(a) (b)

Figure 1. Systematic listing. Correct. (a) Monks problem. (b) Monkeys problem.

Errors could arise when the pupils stopped before they reached the correct answer, as in Figure 2(a), or when they did not maintain the correct ratio, as in Figure 2(b).



(a) (b)

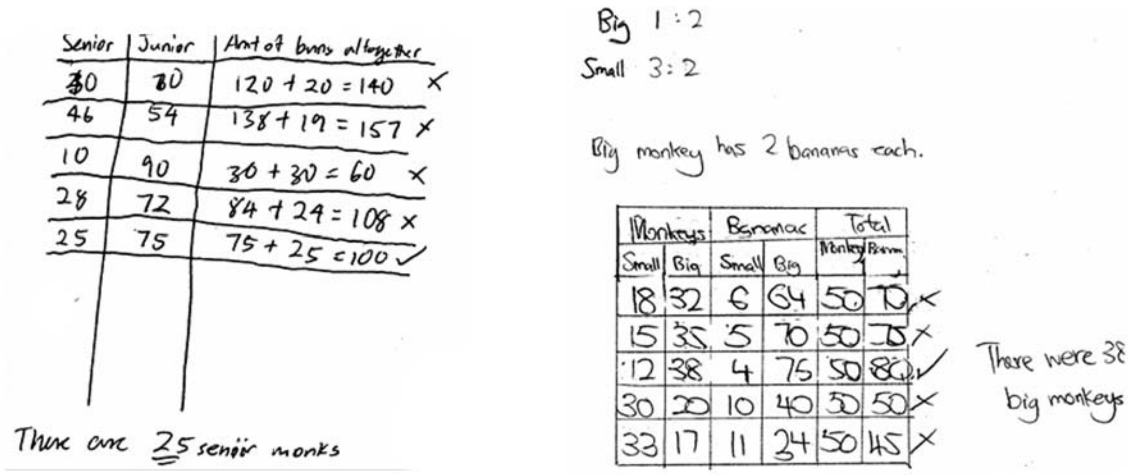
Figure 2. Systematic listing. Partially correct. (a) Monks problem. (b) Monkeys problem.

C2: Guess and Check (Primary: 19, 18; 35, 27; Secondary: 37, 13; 33, 4)

Guess and Check was more popular than Systematic Listing. Secondary pupils were more likely than the primary pupils to get the correct answer with this heuristic. Two examples are given in Figure 3. In Figure 3(b), the pupil divided the number of small monkeys by 3 but did not multiply by 2 (3 small monkeys share 2 bananas). This shows either partial comprehension of the given condition or inability to use proportional thinking.

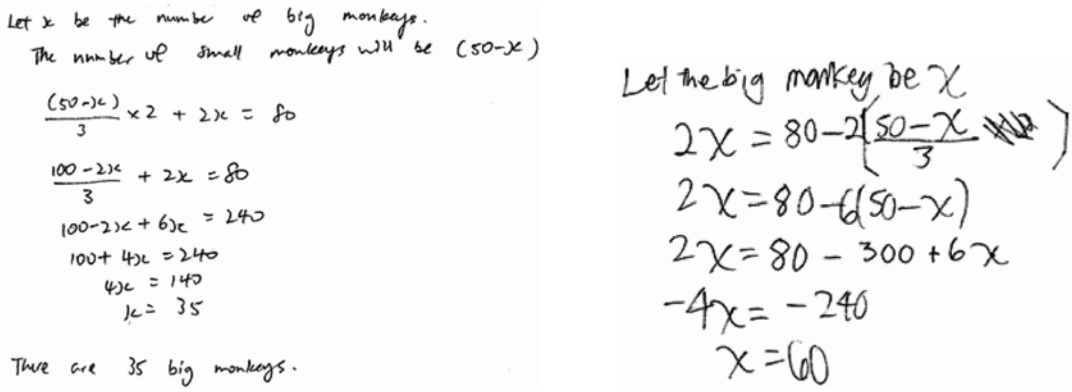
C3: Equations (Primary: Nil; Secondary: 4, 3; 3, 3)

The Singapore primary mathematics curriculum does not include using algebra to solve problems, so none of the primary pupils in the sample used this method. Very few of the Secondary 1 pupils used this method, probably because at the time of the test, they had not learned much algebra. A correct answer is shown in Figure 4(a). The solution in Figure 4(b) contains an arithmetic error about multiplying fraction.



(a) (b)

Figure 3. Guess and check. (a) Monks problem. Correct. (b) Monkeys problem. Partially correct.



(a) (b)

Figure 4. Equations. Monkeys problem. (a) Correct. (b) Partially correct.

C4: Logical Argument (Primary: 26, 7; -, 15; Secondary: 26, 2; 1; 7)

The problem can be solved by forming groups that contain the right ratio. For the Monks problem, a group may consist of one senior monk and three junior monks so that they are assigned to four buns. This is shown in Figure 5, where a diagram was used to illustrate the grouping. Those who tried similar argument were quite successful.

This grouping method does not work readily for the Monkeys problem because the relevant ratio is more complicated, as mentioned earlier. In Figure 6(a), the pupil took “2 bananas” as one share, and the working shows the need to maintain a total of 40 shares in terms of the number of bananas (35 + 5) and 50 monkeys (35 big monkeys and 15 small monkeys). When the ratio is not maintained, the error shown in Figure 6(b) was obtained.

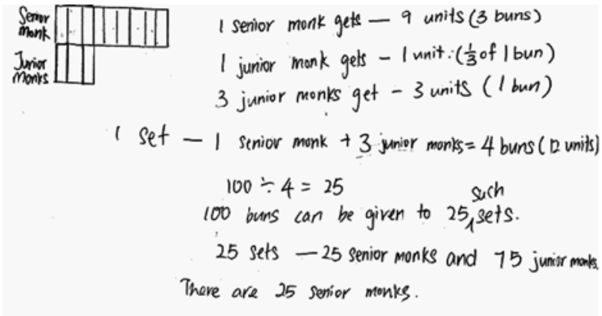
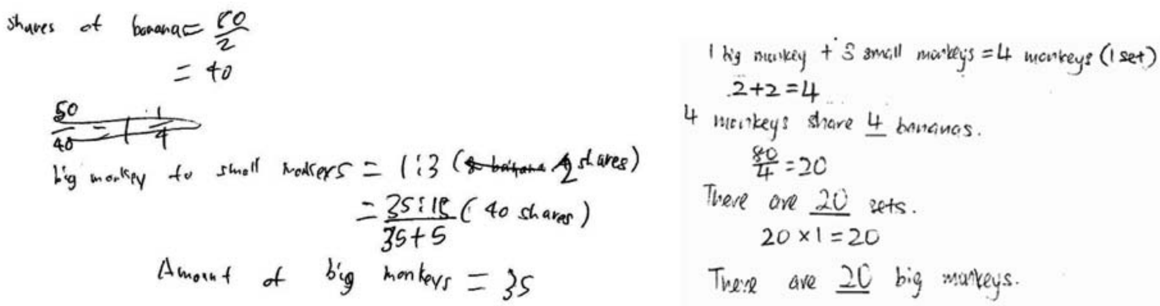


Figure 5. Logical argument. Monks problem. Correct.



(a) (b)

Figure 6. Logical argument. Monkeys problem. (a) Correct. (b) Partially correct.

Other Solutions

The very few attempts to solve these problems using only diagrams were not successful. No pupil wrote down the correct answer without working. About eight pupils in each group re-stated the problems in their own words without making any progress. Many answers could not be coded meaningfully: 23.5% for the Monks problem and 17.2% for the Monkeys problem for the primary group; 13.4% and 2.0% for the secondary group. High percentages of the primary group did not attempt the Monks problem (33.5%) and the Monkeys problem (29.4%), whereas the percentages of no attempts were much lower for the secondary pupils (12.8% and 2.0%).

Inconsistent Use of Heuristics

Table 1 compares the heuristics used by the primary group to answer these two problems. Consider Guess and Check. Forty one pupils used it to solve the Monks problem, 72 used it for the Monkeys problem, but only 27 used it for both problems. This shows that the primary pupils did not use this heuristic consistently to solve parallel problems. However, the success rates were very similar, 46% and 49% respectively. Similarly, inconsistent use of heuristics is also evident for Systematic Listing and Logical Argument.

Table 1

Heuristics Used in Pre-test and Post-test Problem: Primary (N = 221)

		Post-test																		
		C1C	C1P	C1W	C2C	C2P	C2W	C3C	C3P	C3W	C4C	C4P	C4W	C5C	C5W	RS	MW	BLK	ABS	TOT
Pre-test	C1C		1		1															2
	C1P																	2		2
	C1W																			
	C2C		1		**8**	3	1								1	2	1	1	1	19
	C2P		1		10	**2**						2				1		2		18
	C2W					3						1								4
	C3C																			
	C3P																			
	C3W																			
	C4C		1		5	1	2					7	1			1	3	4	1	26
	C4P				3	1											2	1		7
	C4W						1					1						1		3
	C5C																			
	C5W				1	1	1									1	1	1		5
	RS											1				**3**	1	5		8
	MW	1	2		3	7	3									5	**16**	13	2	50
	BLK		2		4	9	2					3					12	**35**	2	74
	ABS																2		**1**	3
	TOT	1	8		35	27	10					15	1		1	13	38	65	7	221

Note: C1: Systematic Listing; C2: Guess and Check; C3: Equation; C4: Logical Argument; C5: Diagram. The last letter means: C = correct; P = Partially correct; W = Wrong. RS = Restating problem. MW = Miscellaneous wrong. BL = Blank. ABS = Absent.

Of the 64 Secondary 1 pupils, 24 used Guess and Check for the Monks problem and 36 for the Monkeys problem, with 21 for both tests. The success rates were 71% for the Monks problem and 89% for the Monkeys problem. For both groups, there was a significant shift to the Guess and Check heuristic.

There were six problems whose solutions could be coded under the same set of heuristics. The total frequency for each code across the six problems was obtained. Note that these frequencies do *not* refer to the *total number of pupils* because a pupil may use the same heuristic more than once when he or she solved the six problems. The findings are provided in Table 2.

Table 2:

Comparison of Heuristics Used in Pre-test and Post-test for Six Problems

Heuristics	**Primary**		**Secondary**	
	Frequency	**Success Rate (%)**	**Frequency**	**Success Rate (%)**
Systematic Listing	99 (111) 63	53.6 (71.2) 54.0	23 (50) 16	52.2 (80.0) 56.3
Guess and Check	66 (72) 38	57.6 (66.7) 39.5	45 (55) 27	84.5 (85.5) 70.4
Equations	79 (68) 52	0 (3.0) 0	80 (63) 39	45.0 (54.0) 46.2
Logical Argument	109 (113) 69	43.2 (30.1) 11.6	133 (112) 83	79.0 (67.9) 61.4
Diagram	65 (54) 24	27.7 (42.6) 45.8	16 (17) 8	93.8 (82.4) 100

Consider Systematic Listing. Among the primary group, there were 99 attempts using this heuristic across the six problems for the pre-test (success rate was 53.6%) and 111 attempts for the post-test (success rate was 71.2%), but only 63 attempts for both tests (54.0%). These values show that the primary pupils did not use

Systematic Listing consistently and their success rates also varied. This inconsistency in heuristic use and different success rates are also observed in the other heuristics. For the primary group, the sizeable number of attempts using Equations was quite surprising, but the lack of success was expected. The secondary pupils also did not consistently use the heuristics to solve parallel problems, although the Logical Argument heuristic was most prevalent, followed by Equations.

Discussion and Conclusion

This paper has provided some evidence to show that upper primary and lower secondary pupils had different success in using five of the common heuristics to solve mathematics problems. This difference may be explained by the interaction between nature of the problems, the pupils' mastery of these heuristics, and the nature of the heuristics. The values in Table 2 indicate that for the types of problems given in this study, Systematic Listing and Guess and Check are quite successful for both groups of pupils. Mathematics teachers might wish to give more attention to these two heuristics. On the other hand, exposing pupils to the different heuristics using pupils' work (perhaps not taken from the same class) adds an authentic touch to promote metacognitive processes in problem solving. Partially correct solutions are particularly useful because the pupils can learn from the mistakes and find ways to improve on the solutions, hence deepening their knowledge of mathematics. The pupils' work can be collected and classified into meaningful codes as a joint effort between teachers and researchers, and this collection will be a useful resource for classroom teaching and teachers' professional development.

A sizeable number of the pupils did not use the same heuristic to solve parallel problems, and the reason for this needs to be examined further. A plausible clue, as indicated in the detailed analysis of the given pair of problems, is that there are subtle differences in the parallel problems that might hinder or facilitate the use of particular heuristics. Another reason might be that these pupils were taught certain heuristics in the months between the two test administrations. In the study, the teachers were not given a copy of the pre-test; hence, the pupils did not have the opportunity to reflect on their methods for the pre-test. This is an issue about research design, namely to reduce the learning effect and to be able to use some of the items for the post-test. However, in real teaching, it matters more to discuss the heuristics used soon after the pupils have attempted the problems because providing timely feedback is a critically effective pedagogical move.

Future research should not rely solely on the analysis of written work. The codes need to be validated through the use of complementary methods such as clinical interviews, error analysis, diagnostic tests, and observations and/or videotaping of pupils solving problems.

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