Neuropsychological Evidence for the Role of Graphical and Algebraic Representations in Understanding Function

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There are difficulties accessing students' thinking about mathematical concepts, although methods such as task observation and interviews provide some useful information. In recent years it has become possible to use functional magnetic resonance imaging (fMRI) techniques to access brain activity while students are thinking about mathematics. In this study we have used this technique to examine brain activity while students were processing graphical and algebraic representations of function. Results show some evidence for increased difficulty of translation between these formats for linear compared with quadratic functions. We also describe regions of the brain that are involved in the translations.

Function is one of the fundamental concepts of school and tertiary mathematics, and yet it is often misunderstood by students and school teachers. This may be due to the number and apparent complexity of its representational manifestations and the concept images these may evoke. In a comparison of the function concept maps of eight professors having PhDs in mathematics with those of twenty-eight university mathematics students, Williams (1998) found the latter had an emphasis on minor detail, such as the variable used, algorithms, and the idea that functions are equations in the student maps. In contrast she found that "none of the experts demonstrated the students' propensity to think of a function as an equation. Instead, they defined it as a correspondence, a mapping, a pairing, or a rule." (*ibid,* p. 420). Some, including teachers, have a tendency to think of functions graphically and in terms of processes, even to the extent of separating algebra from functions (seen as graphical) in their thinking (Chinnappan & Thomas, 2001), rather than seeing function as a concept crossing representational boundaries. Even's (1998) research with college students exemplified the importance of representations in understanding of function, with students having difficulties flexibly linking different representations and finding links between pointwise and global approaches to function problems.

Mathematics is essentially a symbolic practice in which signs are used, invented, and re-created (Saenz-Ludlow & Presmeg, 2006), and in particular one where we have the capacity to substitute some signs for others (Duval, 2006). Hence it is important to have a semiotic perspective on the role of signs in mathematical learning and practice (Radford, 2000). The situation is complicated by the relationship between the external sign and its internal interpretation by the individual, which is necessarily a function of the individual's existing cognitive structure. Thus, the word representation is used in different ways in the literature, either to refer to external signs that are part of a system or for their cognitive analogue. In this paper we will try to maintain a distinction between the use of the word sign for the external stimulus and representation for the internal, cognitive construct. One of the key aspects of mathematics is that it is probably the knowledge domain in which we find the largest range of semiotic representation systems (Duval, 2006), and hence its concepts need to be understood via a multiplicity of signs, (and hence representations) with each sign emphasising or de-emphasising different characteristics of a concept. As Otte (2006) notes, this involves an epistemological triangle between the sign, its referent (and hence representation) and the conceptual object, with epistemology involving the relationship between such entities, objects and signs. Thus developing rich mathematical thinking requires the ability to establish meaningful links between representational forms and translation from one representation of a concept to another, which Thomas (2008), calls *representational versatility,* a construct that includes qualitatively differing cognitive interactions with signs, through representations. Duval (2006) describes translations between systems as *conversions*, and it is the challenging nature of some conversions that makes some mathematics so demanding for learners. For example, the concept of function has associated graphical, algebraic, ordered pair, tabular, and other representation systems, with the links between each of them contributing to overall understanding of the concept.

Since 1991 brain function and cognition has been studied using functional magnetic resonance imaging (fMRI). This employs nuclear magnetic resonance (MR), which is non-invasive and produces images of the human body with excellent soft tissue contrast. Areas of the brain that become active show a temporary increase in blood supply, and the resulting change in the ratio of oxygenated to deoxygentated haemoglobin, can be measured by fMRI using the blood oxygen level dependent (BOLD) contrast. While most mathematical experiments involving imaging techniques have considered the most elementary of concepts, such as number, counting or arithmetic (e.g., Butterworth, 1999; Dehaene, 1997), its use in other mathematical investigations has been relatively rare, probably due to the difficulties inherent in experimentally isolating higher cognitive processes. One exception is the use of fMRI to study learning of elementary algebra, such as equation solving in highly competent college students (Anderson, Qin, Sohn, Stenger, & Carter, 2003), where solutions to linear equations were categorised into three levels of complexity by the number of transformations required to solve them. Results showed that the size of the BOLD response in the parietal and prefrontal regions directly reflected the number of transformations occurring. A later study (Qin, Carter, Silk, Stenger, Fissell, Goode, & Anderson, 2004) compared results using the same kinds of equations with ten 12–15 year old students to those from the previous study with adults. They found that the active areas in children's algebra equation learning were similar to areas active in adults, except that in children the parietal cortex showed a practice effect that was not found in adults. Recent research by Lee et al. (2007) considered the role of problem representation. They compared model (lengths of rectangular boxes represent variable values) and symbolic methods for construction of linear equations from word problems. They concluded that there were more accurate responses in the model than in the symbolic condition. One reason for the efficacy of the model method was its lower demand on attentional resources. There have been virtually no fMRI experiments involving mathematical concepts beyond basic algebra. In this research we were concerned with using fMRI to investigate whether one can identify brain areas active in the process of translation between graphical and algebraic function representation systems, or registers, if this brain area activity is format dependent or independent for graphical and algebraic representational formats, and whether the translation is independent of format direction.

Method

The ten participants in the study comprised five undergraduates ('novices") and five graduates ('experts") at The University of Auckland. There was no significant difference in age between the experts and novices. Gender was balanced (but not across the expertise groups which were eventually collapsed). The study was approved by the local ethics committee. All participants received instructions and training on a subset of stimulus items in an initial screening session. They also filled out a questionnaire at this session, including the question "How often do you think of mathematics in terms of pictures or images?". Several days later they returned for an fMRI session and were scanned for 40 minutes while carrying out simple mathematical tasks involving function representations. These tasks required viewing pairs of mathematical functions (graphs or equations) that were flashed consecutively on a computer screen above their head while they lay in the MRI scanner. For each pair they had to press a button to indicate whether the functions were the same or different. There were four different experimental conditions considered, two "same format" conditions where participants were asked whether the stimuli represented the same function (graph to graph and algebra to algebra), and two "cross format" conditions (graph to algebra, and algebra to graph), with the same question. The functions were presented for 200 ms each, with a blank screen between, with a total of 6000 ms for each trial. Participants responded by pressing a key with their left index finger for "non-matching" or a key with their right index finger for "matching". The stimulus pairs were presented in short blocks of five differentiated by equation type, linear versus quadratic. There were four of these sequences altogether giving a total of 20 stimulus pairs in each condition. Brain activation was compared to a baseline condition where particpants fixated on a small central cross with their eyes open. Experimental software recorded response accuracy and reaction time (RT). Participants completed a post-experimental questionnaire in which they were asked to rate the exent to which they employed various strategies, and were given an open question about strategy use.

Results

Analysis revealed no differences in performance between the novices and experts so results here represent a single group. Reaction times and accuracies[11] showed that the cross-format conditions (graph to algebra, algebra to graph) were significantly more difficult than the non cross-format conditions (graph to graph, algebra to algebra). Overall, participants were 543 ms slower responding to the cross-format questions, and

11 Some data from one participant is missing in this analysis due to experimenter error.

had an average accuracy of around 84%, compared to 93% for the same format conditions (see Tables 1 and 2 for representative examples of questions with corresponding accuracy). This effect of difficulty was slightly larger for the novices, but was not statistically significant for the small sample size of nine. The two cross-format conditions did not differ significantly in accuracy or RT, nor did the two same format conditions. While there are other, possibly significant, interactions present, notably with the within/cross format effect mentioned above, However, it is instructive to analyse this difference to attempt to understand the relative roles of the graphical and algebraic representations for each function type.

One might expect that translation of quadratic equations would be more difficult than linear ones, but they proved actually slightly easier overall, with a more accurate response by participants to the quadratic than the linear functions in both directions of translation (graph to algebra: 90% quadratic vs. 79% linear, respectively, $n=9$, Wilcoxon $z=2.25$, $p<0.05$; algebra to graph: 89% quadratic vs. 79% linear, respectively, Wilcoxon $z=2.18$, $p<0.05$), although there was no significant difference in RT between the two, with a combined mean time of 1376 ms for the graph to algebra condition, and 1317 ms for the algebra to graph condition.

Table 1

Representative Questions for the Blocks Involving Translation Algebra to Graph

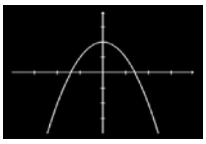
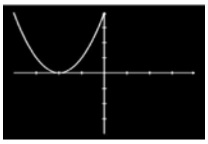
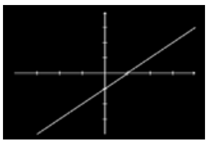
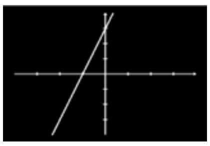
First stimulus	Second stimulus	Item Numbers	Correct Response	Correct
$y = -x^2 + 2$		13, 13	True	100%
$y = (x - 2)^2$		8, 10	False (Translation wrong direction)	67%
$y = x - 1$		5, 5	True	88%
$y = -3x + 3$		19, 17	False (Gradient wrong)	50%

Table 2

Representative Questions for the Blocks Involving Translation Graph to Algebra

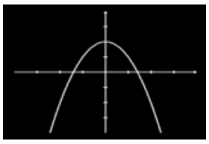
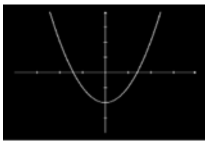
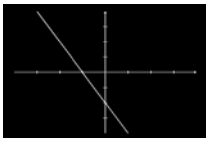
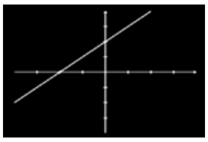
First stimulus	Second stimulus	Item numbers	Correct response	Correct
	$y = -x^2 - 1$	13, 15	False (Translation wrong direction, size)	100%
	$y = x^2 - 2$	5, 5	True	88%
	$y = -x + 2$	16, 10	False (Wrong slope, intercepts)	63%
	$y = x + 2$	3, 3	True	78%

Table 3 shows a breakdown of accuracy based on the individual items, allowing a comparison of direction of translation between linear with quadratic functions. In this analysis we find that the translation from graph to algebra was significantly more difficult for linear than for quadratic functions (t=2.27, p<0.05). However, when the direction of translation was reversed the difference in accuracy was not significant. It is clear from Table 3 that there was no difference in difficulty for linear function items when translating from algebra to graph compared with graph to algebra, and the same was true for quadratic functions. This is contrary to the finding of Duval (2006), who suggests that recognition of linear graphs, measured by conversion of graphical sign (register) was relatively poor (25% success for y=2x) compared with conversion of an algebraic sign (register).

Table 3

Mean (SD) Translation Accuracy Averaged Across Linear and Quadratic Function Items

Direction of Translation	Linear (*N*=20)	Quadratic (*N*=20)
Algebra to Graph	81% (20%)	89% (16%)
Graph to Algebra	78% (21%)	90% (12%)

How can we explain the fact that the linear translations in either direction were apparently more difficult? In analysing the results of this condition one must ask: what does one need to notice or pay attention to (Mason, 2003) in order to be able to respond to whether the algebraic equation matches the graph? Two aspects on which attention can be focused are perceiving specific properties and reasoning on the basis of these properties that are taken to characterise objects. For the linear functions, corresponding to $y = \pm mx \pm c$, $m, c \in \{-3, -2, -1, 0, 1, 2, 3\}$, working from graph to equation one would expect that the salient features, or properties, of the graphs to remember were: whether it had a positive or negative gradient; the value of the x-intercept; the value of the y-intercept; and hence the value of the gradient (see Tables 1 and 2 for examples). This last piece of data is a calculated value, rather than a perceived one, requiring a procedural interaction with the graphical representation rather than observation (Thomas & Hong, 2001). When working from the algebra to the graph the functions were all given algebraically (see Tables 1 and 2 for examples), and so one might note: the gradient from the first $\pm$; the y-intercept from the c, and the gradient from the m. This analysis, which shows that more data needs to be analysed in translating from graph to algebra than viceversa, agrees with that of Duval (2006), who provided evidence that conversion from graphical to algebraic format for linear functions proved harder for students than going in the opposite direction.

For quadratic graphs, in the graph to algebra condition one again needs to pay attention to the salient features of the graph of the quadratic function. The participants had all practiced the task and so were familiar with the types of function used, namely simple translations of $f(x)=x^2$, of the type $\pm f(x)\pm k,\ k\in\{1,2,3\}$ or $\pm f(x\pm k),\ k\in\{1,2,3\}$. The use of $f(x)=x^2$ removed the added complication of needing to find the gradient multiplier (as in $f(x)=kx^2$) and hence, one needs to see just four things: the orientation of the graph to obtain the first $\pm$; whether it is translated parallel to the *x*- or *y*-axis; the direction of the translation for the second $\pm$; and its size (see Tables 1 and 2 for examples of the graphs). In the algebra to graph condition the participants were presented with an algebraic equation (see Tables 1 and 2 for examples). Again one needs to note: the first $\pm$ giving the orientation; the kind of translation; the direction of the translation from the second $\pm$; and its size.

Questionnaire Results

Examining the questionnaire data can give a measure of the validity of the above analysis. In answer to the question How often do you think of mathematics in terms of pictures or images?, 70% said quite often or most of the time, compared with 20% responding rarely or not very often. Table 4 shows the percentages responding with Agree (A)/Strongly Agree (SA) or Disagree (D)/Strongly disagree (SD) to the other questions.

Table 4

Strategy Question Facilities (N=10)

Question	**% A/SA**	**% D/SD**
Graph to algebra		
As soon as I saw the graph I started to put it into equation form	50%	40%
I focused on trying to keep a picture of the graph in my head	50%	40%
As soon as I saw the graph I tried to pick out key aspects	100%	0%
I didn't need to have a strategy it was obvious	0%	80%
I'm not really sure which strategy I used	10%	80%
Algebra to graph		
As soon as I saw the equation I started to imagine the graph	90%	0%
I focused on trying to repeat the equation to myself	30%	50%
As soon as I saw the equation I tried to remember key aspects	100%	0%
I didn't need to have a strategy, it was obvious	10%	60%
I'm not really sure which strategy I used	0%	80%

This data gives a reasonably clear picture of what the participants' strategies were. The majority (80%) in each case were clear about their strategy, and all of them responded that their method was to try and remember 'key aspects' or properties of either the graph of the equation they were shown, and then match these properties to the next image. While doing this some tried to keep the graph (50%) or the equation (30%) in mind, but the others were clear they did not do this. We note too that while the equation evoked an immediate attempt to see the graphical representation for 90% of the participants, but the graph was treated in this manner by only 50%, giving some evidence of a stronger visual link for algebra to graph than for graph to algebra. The open responses, particularly from the 'experts' confirmed the role of the property search strategy we have described above. For the direction graph to algebra S1 responded "Sometimes I would see the graph, then see the equation and tried to 'see' the graph again. But in general I would say that I looked for key aspects of the graphs first." Other comments were:

> S2. If possible, I worked out the equation for the graph in the gap and then checked it against the equation that came up. I did this by picking out the key features, like intercept, translation, positive or negative slope etc. that determine the equation's form. When the equation came up on the screen I also did the reverse.

S7. When the equation came up I would not take in the whole equation (for example…I wouldn't think about $y = x^2 + 1$ but look for corresponding details i.e. $-x$ or x and number at the end i.e. 1, although I think that often with the adding number I wouldn't pay so much attention to the sign in front of it. So it was more a matter of the form of the equation.)

For the translation from algebra to graph direction they said:

S2. I looked at the graph and worked out key features, which meant I could work out what the graph would look like. I then compared it to the graph shown. I think when I compared the two graphs I didn't so much compare them visually, but I compared there [sic] key features.

S7. Like the other case I'd generally try pick out details i.e. presence of negatives or added constants and then find corresponding bits on graph in a general sense as I had certain expectation of what those parts would combine to give in overall shape and slope.

It is clear that these participants were using the key properties of the signs, there is also a sense from S7 that he was concerned about the overall form or structure of the sign. We will be reporting in more detail about the areas of brain activity seen and conclusions we can draw from these in a further paper but we present here several results. Significantly greater activity for graphs versus equations was seen in right superior parietal cortex (previously associated with mathematics and attentional shifting), in visual areas in the occipital cortex (left mid and right superior), and in the right mid temporal cortex (possibly associated with object processing). Whether this is due to mere perception of the stimulus as opposed to higher-level processing is hard to know. The extent and amount of brain activation was much greater when participants had to translate between formats (see Figure 2), and activation specific to these cross-format conditions was seen in the left inferior frontal gyrus (Broca's area; related to speech production and verbal memory), and two small clusters in the right hippocampus (associated with memory) and right cingulate (associated with cognitive control/attention).

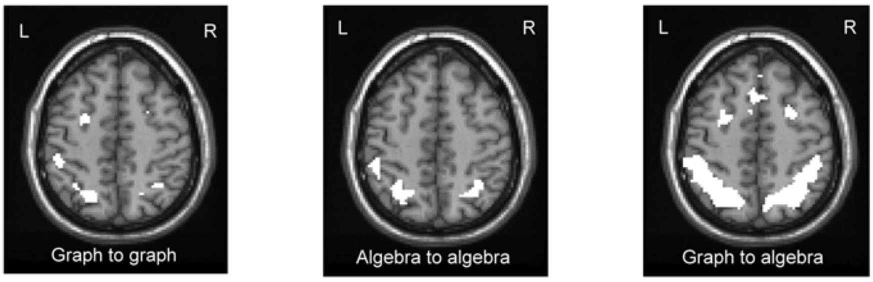


Figure 1. The extent and amount of brain activation for within and between graph and algebra formats.

Thirdly, we considered whether there might be brain areas involved in format independent representation of functions. In this case the intersection of active areas for within format comparisons between the two conditions should include these areas. We found significant conjunction clusters in an area already associated with abstract representation of number, the intraparietal sulcus (IPS), as well as other nearby areas often associated with mathematics, the superior parietal lobule (bilateral), and the left angular gyrus. Smaller clusters were also found in the left mid frontal cortex and right lingual gyrus. Some of these areas (most likely the IPS) may be related to format independent representation of functions, although this requires confirmation in future experiments.

Conclusion

What are the possible implications of these results for the teaching and learning of function? The concept of *representational versatility* (Thomas, 2008, p. 79), includes "*the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations*", that is, it encompasses both the treatments and conversions of Duval (2006). One point arising from this research is that one should not assume that translations (or conversions) of linear functions are easier for students to process than quadratic functions, since for this sample they were more difficult, in both directions of translation, probably due to the increased cognitive load of having more properties to pay attention to. In contrast Duval (2006) describes evidence that the conversion from graphical to algebraic format for linear functions proved harder for students than going in the opposite direction. However, we agree that "The true challenge of mathematics education is first to develop the ability to change register." (*ibid,* 2006, p. 128). Thus to learn which changes in function representations are significant students need to work at translating between representations. Unfortunately, there has been a tendency on the party of some to separate the teaching of algebraic, or equation, function signs from their graphical counterparts, making the likelihood of such translations very small. The translation difficulties we have discussed above may suggest that students would find the relationships between the function signs easier to comprehend, and the properties easier to pay attention to, if their teaching was integrated for each function type and tasks were set that encouraged students to make explicit links between the properties of each sign, in the context of an overall structure.

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