What Does Three-quarters Look Like? Students' Representations of Three-quarters

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Forty-one students from Years 3 to 10 completed a clinical interview on the topic of fractions. Four of the questions from the 20-30 minute interview involved the fraction $^3/_4$; students' responses to these questions are analysed in this paper. When asked to illustrate $^3/_4$, the most popular model chosen by students was a circle, yet fewer than 25% of the students knew that a circle divided into 4 parts with unequal areas (using 3 vertical lines) did not represent $^3/_4$. Such students are unable to identify the appropriate attribute which is relevant to the given model.

Understanding fraction concepts is clearly difficult for many students. For example, 69% of the Year 8 Australian sample in TIMSS (2003) correctly chose $^{16}/_{30}$ on this multiple choice item: *In a group of children, 16 have birthdays during the first half of the year, and 14 have birthdays during the second half of the year. What fraction of the group have birthdays during the first half of the year?* (Item Number M012041). Although this result compares favourably with the international average (52%) it is clear that about one in three Australian Year 8 students in this sample lacked fundamental fraction understanding after about four years of the topic being included in the school curriculum. This paper will further the research into students' understandings of fractions by investigating the models they use to represent fractions. In addition, we wish to determine if students understand the essential features of the models they use.

The interviews upon which this paper is based are part of a larger research project investigating the strategies that students use to compare fractions and how their choice of strategy is influenced by their understanding of fractions. The interviews included a mixture of tasks: for example, a fraction symbol was provided and the student was required to draw a representation either on a blank card or on the diagram/shape provided; the student needed to select which of the given representations matched the spoken fraction word name; and students needed to reconstruct the whole, given the part. The interviews also included a range of representations: discrete models and continuous models in various dimensions (linear, area, and drawings of volume), as well as ratio, operator, and quotient interpretations of fractions. The items considered in this paper all involved the fraction ¾.

Literature

Lamon (2007) provides a comprehensive discussion of research into rational numbers conducted over the past two decades, and the various interpretations which have been used: that is, part-whole, measure, operator, quotient, ratio/rate. Lamon notes the difficulties in teaching and learning topics involving multiplicative structures:

Of all the topics in the school curriculum, fractions, ratios and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites. (p. 629)

The teacher of the Grade 6 students in the study of Olive and Vomvoridi (2006) assumed that students were aware of the need for equal sized partitions of the whole (8 pizzas shared among 10 people) and did not attempt to draw this or mention it explicitly. The teacher's "use of approximate representations throughout her instruction, coupled with the lack of any explicitly spoken intention to draw equal parts, may have unintentionally supported some students' lack of an equipartitioning scheme for unit fractions" (p24).

Various researchers have noted the relative difficulty that many students experience with number line tasks. For example, Hannula (2003) reported on two tasks involving $^3/_4$ with over 1000 students in each of Grades 5 and 7. In the bar task, students were provided with a rectangle/bar (already marked in eighths) and asked to shade ¾. In the number line task, students were provided with a number line where 0 and 1 were marked, but were not the endpoints. As well as noting improvements in performance with grade, he found that the (overall) success rate for the number line task was 38% in contrast with 71% success in the bar task. Clarke, Roche, and Mitchell (2007) also found students experienced difficulty with tasks involving number lines. They report on clinical interviews with over 300 Grade 6 students and found that only around half of the students could draw an appropriate number line that showed $^2/_3$. Bright, Behr, Post, and Wachsmuth (1988) noted that students in their study found the 0 to 1 number lines easier than the 0 to 2 number lines.

Baturo (2004) devised a Cognitive Diagnostic Common Fraction Test to probe students' fraction understanding. The first item contained various two-dimensional drawings, some of which were equipartitioned but unusual shapes (e.g., a rhombus) while others were typical fraction shapes (e.g., a circle) that were not equipartitioned. Responses to such diagrams allow us to glimpse into students' thinking. Students who accept (incorrectly) all familiar shapes are not attending to need for equipartitioning. In contrast, students who reject (incorrectly) all unfamiliar shapes appear to be making decisions based on their familiarity with the shape; that is, they recognise only prototypical or iconic images.

Out study continues the tradition of clinical fraction interviews and includes tasks to probe students' understanding. We also note the cautions expressed by various researchers: that correct answers to one fraction task does not imply success on others and, that students can have difficulty verbalising their thinking. For these reasons we included complementary tasks to probe students' thinking with an expectation that some of these could be adapted to other modes of assessment (for example, pen-and-paper or online tests) having a strong diagnostic emphasis.

The sample

The 41 interviews reported in this paper were conducted as part of a larger project that involved testing approximately 800 students from Years 3 to 10 in four schools to determine the prevalence of various incorrect strategies for ordering fractions. The researchers compiled a list of students who made at least one error on the test as potential interviewees. Schools were provided with these lists and approximately 50% of these students returned signed consent forms; return rates were much lower for the older students, compared with the younger students. The 41 students who were interviewed do not constitute a random sample at each year level; rather, they have been chosen from the consenting students to create a diverse group of "non-expert" students. This procedure was used to ensure that interview time was spent only on non-expert students who could contribute to our knowledge of incomplete understanding. Table 1 contains the number of students interviewed at each year level.

Table 1

Number of Students Interviewed by Year Level

Year level	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7	Yr 8	Yr 9	Yr 10	Total
Numb er of students	9	9	6	5	6	1	3	2	41

The Interview

The audio-taped interviews lasted 20 to 30 minutes and were conducted by the researchers in a quiet location in each school. Each student completed approximately 25 tasks, but only four tasks are discussed here. These tasks involved the fraction three-quarters, a non-unit fraction which students were expected to be familiar with, and which is easy to represent by repeated halving (hence avoiding the issues associated with partitioning into three or five equal parts). We wanted to learn more about students' conceptions of fractions, including the flexibility to move between various representations. As indicated in Table 2, in three of these tasks the students were handed a single card and the written instruction was read aloud by the researcher. In the remaining task, the students were handed a set of 8 cards and given a verbal instruction only. The four tasks are described in more detail below.

Task 6, *Model of choice*, is an unpublished item used in the interviews reported by Pearn (Pearn et al., 2003). The symbol $^3/_4$ was written on a card and students were asked to "draw a picture or diagram to show what it means" as well as "explain" their drawing. Student's drawings were judged to be correct or not, as well as analysed according to their chosen representation. This task was given early in the interview so that students' responses to this task would not be affected by the representations contained in other interview tasks.

In Task 8d, *Unit square* (Hollis, 1984), an unmarked square printed on a card and students were asked to "shade the shape to show the fraction $^3/_4$". Students' drawings were analysed for correctness as well as according to the way they subdivided the square.

In Task 9, *Sorting* (Willis, 2004), students were handed a set of 8 cards (labelled A to H) and given this verbal instruction: "Some of these cards show the fraction three-quarters and some don't. Sort them into two piles; put the cards showing three-quarters here and the other cards there". Details of Cards A to H are provided later in Figure 1, accompanied by the results. Cards A and D were prototypical representations of $^3/_4$ using the circular (area) and discrete models, respectively. Cards B and H were included to determine whether students appreciate the need for portions of equal area. Cards F and G were included to determine whether students had over-specialised their interpretation of $^3/_4$; they would reject F if they felt that the portions need to be adjacent and reject G if they did not understand equivalent fractions.

An additional task that also included the fraction $^3/_4$ was Task 16, *Number line*, (Newstead & Olivier, 1999). Students were handed a card with a number line with marks on 0, 1, 2, 3, and 4, and a list of four numbers; (A) $^3/_4$, (B) $1^1/_3$, (C) $^{12}/_6$, and (D) $^2/_3$. Students were asked to "Show the following numbers on the number line".

Table 2

Details of the Four Interview Tasks

Task Number & Name		Number of cards	Instructions	Symbol $^3/_4$ provided?	Representation provided?
6	Model of choice	1	Written & verbal	yes	no
8d	Unit square	1	Written & verbal	yes	Unit square
9	Sorting	8	Verbal only	no	8 models
16A	Number line	1	Written & verbal	yes	Number line

Results

The number of students completing each of the four tasks is provided in Table 3, as well as the number and percentage of students correct on each task. Clearly the first two tasks (where students were required to draw their own model to show $^3/_4$, and then to show $^3/_4$ of a unit square) were much easier for students than Task 9 (involving sorting a set of 8 cards) and Task 16A (marking $^3/_4$ on a 0 to 4 number line).

Table 3

Number and Percentage of Students Correct on Each of the Four Interview Tasks

Task Number & Name		Number of students who completed task	Number of students correct	Percentage correct given completed
6	Model of choice	40	32	80%
8d	Unit square	40	36	90%
9	Sorting	41	4	10%
16A	Number line	35	16	46%

Task 6: Model of Choice

Task 6 was answered correctly by 80% of the 40 students who completed the task. The most common correct response (17 students out 32) was a circle partitioned into four parts which appeared to be roughly equal, using one vertical and one horizontal line to create four sectors, where three parts are shaded. Another 7 students used the same partitioning on a square. Of the remaining 8 correct students, 4 students drew a rectangle (with base longer than height) which was partitioned with three vertical lines, and the other 4 students used a discrete model and shaded three of four objects. There were only 8 incorrect responses to this task; almost all of the incorrect responses came from students in Years 3 and 4. The most common incorrect response was to show 3 groups of 4.

Task 8d: Unit Square

Of the 36 students who gave correct answers to Task 8d, 32 students (89%) used the prototypical partitioning of the square (with one vertical and one horizontal line); three students used vertical partitioning (using three lines) and only one student used two diagonal lines.

Task 9: Sorting

All 41 students completed this task. Whereas only 4 students (10%) completed the full task correctly, over 80% of the students correctly placed 4 of the 8 cards (Cards A, D, F, and C). Clearly some of the cards were more difficult for students to sort correctly. The number and percentage of students who answered correctly on each of the 8 cards is provided in Figure 1. The asterisk on some cards indicates that they do *not* represent $^3/_4$ and so the correct response is to *reject* the card. The cards are presented in decreasing order of student performance. Card A was the easiest (40 students recognised that Card A does represent $^3/_4$) and Card B the hardest (only 10 students recognised that Card B did *not* represent $^3/_4$).

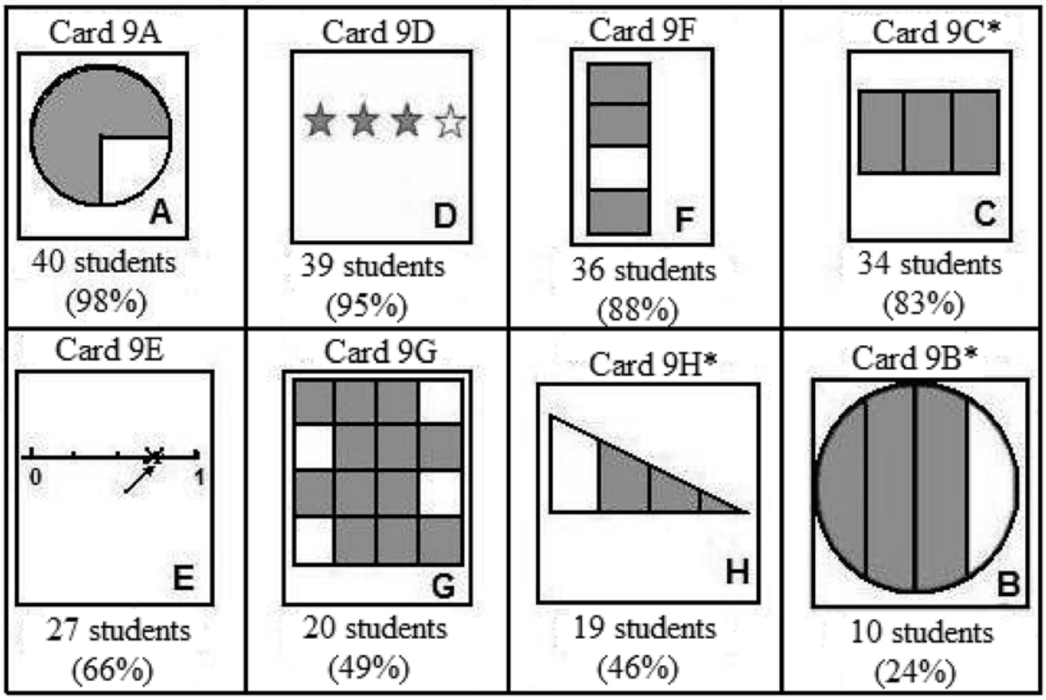

Figure 1: Number (%) of students correct on each card in Task 9 (*Sorting*), ranked from easiest to hardest. (* indicates correct answer is not $^3/_4$)

This group of non-experts was able to demonstrate the following fraction understandings: that the number of portions need not match the numerator (there is only 1 shaded portion in Card A); that fractions can be represented with a discrete model (Card D, although this is a special case where the denominator matches the number of elements in the set); that shaded portions do not need to be contiguous (Card F), and that matching the numerator with the number of pieces is not sufficient (Card C). In comparison with the discrete model (Card D), only two-thirds of the 41 students were able to recognise the number line model (Card E).

About half of the students accepted Card G as showing $^3/_4$. Of those who rejected this card, some might have made “minor” errors, such as miscounting or miscalculating when determining equivalent fractions, but it is likely that for many of these students, the rejection of this card indicates that the concept of equivalent fractions is not understood. All the students who talked about rearranging the squares in some of the rows (e.g., the second and fourth rows) successfully identified this card as showing three-quarters from the fact that the three shaded squares in each row could be seen more clearly.

Cards H and B were the most difficult cards for students. Note that the correct response for both cards is that neither represent $^3/_4$, and hence should be rejected. (While it might appear that this could be the reason that students found these more difficult than the other cards, consideration of Card C disproves this hypothesis as it also does not represent $^3/_4$, yet was answered correctly by over 80% of the students.) To further investigate students' responses to Cards H and B, Table 4 provides a cross-tabulation of the 41 students who completed this task. Only 8 students were correct on both cards; their correct rejection indicates that they appreciate the requirement for equal-sized portions when using the area model.

Table 4

Cross-tabulation of Responses to Cards H and B in Task 9 (Sorting)

Card H*	Card B*		Total
	Correct	Incorrect	
Correct	8	11	19
Incorrect	2	20	22
	10	31	41

* The correct answer is to reject these cards as showing $^3/_4$

Another 20 students were incorrectly accepted both cards as representing $^3/_4$. These students may be focussing on the number of shaded parts (3) and the total number of parts (4) without attending to the need for equal-sized portions, or they may focus their attention on the equal-sized *lengths* (on the base of the triangle in Card H, and the diameter of the circle in Card B). In retrospect, additional cards should have been included which had unequal partitions in length (i.e., one-dimensional) as well as the unequal partitions in area (two-dimensional). Table 4 shows that another 11 students were correct on Card H but incorrect on Card B; so whereas they rejected Card H, this was not done by a careful consideration of the unequal areas (as this reasoning should have resulted in correctly rejecting Card B). Their reason for rejecting Card H and accepting Card B might be that the triangle is not a familiar model for fractions, while the circle is a familiar model. In other words, for some students, their familiarity with the shape in the context of fractions seems to be an important issue, rather than a consideration of the essential features of fractions.

To further contrast the success rate of 80% on Task 6 (Model of choice) with the much lower rate of 24% on Card B in Task 9, students' responses to the two tasks were examined. Of interest are the 11 students who chose to use a circular model in Task 6 and correctly drew a prototypical representation of $^3/_4$ but were then unable to reject the circle with four equally spaced vertical partitions on Card B in Task 9. These 11 students were spread across Years 3 to 9 indicating that this is not solely a difficulty of younger students.

Task 16A: Number Line

Table 3 shows that only 16 students (46% of the 35 students who completed this task) were able to correctly locate $^3/_4$ on a number line from 0 to 4. There were two common wrong answers; 7 students marked the point 3 (they were clearly finding $^3/_4$ of the given line, so seeing $^3/_4$ as an operator rather than as a point which is less than one) and another 7 students who marked $^3/_4$ somewhere between 3 and 4. Figure 2 is a sample from a Year 5 student who marked $^3/_4$ at 3 and then correctly marked the next three points; inconsistently marking $^2/_3$ as a point (measure) rather than as an operator. Perhaps, after marking both B ($1^1/_3$) and C ($^{12}/_6$) as measures rather than as operators, this student continued with this interpretation for D ($^2/_3$). These results confirm Hannula's (2003) findings. He also found students who correctly marked certain numbers on the number line (e.g., 1.5 and $2^1/_5$) but were unable to mark $^3/_4$. He noted that some students could not make any mark as they thought that $^3/_4$ was not really a number. For those that did make a mark, some placed $^3/_4$ at 3.4, and others used the operator interpretation and attempted to find $^3/_4$ of some unit, which may have been the full length of the line provided, or some sub-interval.

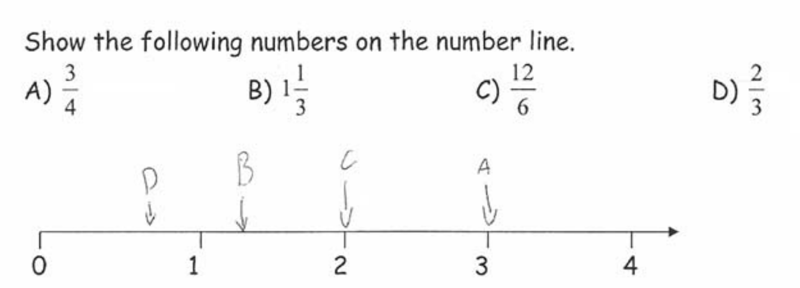

Figure 2: Sample of Task 16 (Number line) by a Year 5 student. Note the placement of ¾ at 3.

It is interesting to compare students' responses to the two tasks with number lines. Of the 25 students who could recognise $^3/_4$ on a 0 to 1 number line (Card E in Task 9), and who also completed Task 16A, only 12 were able to reproduce this point on the longer number line. We also note that two groups of students (i.e., students with measure or operator interpretations of fractions) will answer correctly on the 0 to 1 number lines, but only the students with the measure interpretation will answer correctly on longer lines.

Conclusion

When asked to illustrate $^3/_{4,}$ the most popular model chosen by students was a circle, yet fewer than 25% of the students knew that a circle divided into 4 parts with unequal areas (using 3 vertical lines) did not represent $^3/_{4.}$ Hence, we have found evidence of students who recognise only the prototypical or iconic representations of fractions. These students might initially appear to demonstrate fraction understanding, however their tendency to accept incorrect models means that they are not paying attention to the essential features of fractions. When students were asked to explain their thinking during the interview (for example, on the sorting task) students typically mentioned "three out of four parts are shaded" and did not discuss equal lengths or equal area. This is an instance where it might be possible to diagnose their thinking from their actions rather than their words.

The confusion between *3 groups of 4* and *3 out of 4* was found to be limited to students in Years 3 and 4. About two-thirds of the students recognised the point $^3/_4$ on a 0 to 1 number line, and about half of this group was later able to mark a longer number line with the point $^3/_4$. Clearly some students confuse the operator construct with the measure construct, and further research might confirm that asking students to mark numbers greater than 1 (such as $1^1/_3$ and $^{12}/_6$) helps them to focus on the measure construct.

The focus of this paper has been an analysis of students' representations of the fraction $^3/_4$. The sample of 41 students from Years 3 to 10 is not intended to be a representative sample, but rather a collection of non-expert students whose differential abilities to complete various tasks has confirmed results of other research and included some surprises. On the other hand, these non-expert students were not hard to find, so we can expect that there will be some students in most classes from Years 7 to 10 with fraction knowledge that is confused, incomplete, or incorrect and who exhibit basic misunderstandings.

We concur with Clarke et al. (2007),

> Students need more opportunities to solve problems where not all parts are of the same area and shape … it is clear that fraction as a measure requires greater emphasis in curriculum documents and professional development programs, as many students are clearly not viewing fractions as numbers in their own right" (p. 215).

Students need to have their attention directed to the *attribute* on which the model is based: relative number (for the set model), relative length (for linear or one-dimensional models), relative area (for two-dimensional models), and relative volume (for three-dimensional models). Without this attention, students are merely trying to recognise familiar pictures or icons.

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