The Identification of Partially Correct Constructs

Gila Ron
Tel Aviv University, Israel
<gilaron@bezeqint.net>

Rina Hershkowitz
Weizmann Institute of Science, Israel
<rina.hershkovitz@weizmann.ac.il>

Tommy Dreyfus
Tel Aviv University, Israel
<tommyd@post.tau.ac.il>

We show how the RBC model for abstraction in context can be used to follow the emergence of a learner's knowledge constructs and to identify in detail the learner's partially correct constructs (PaCCs). These PaCCs are used to explain the learner's inconsistent answers and provide added insight into processes of knowledge construction. The research process is illustrated by means of an example from elementary probability. We thus demonstrate the analytic power of the RBC model for abstraction in context.

Abstraction has been a central issue in mathematics and science education for many years. The classic work by Piaget, Davydov, Skemp and others has in recent years been succeeded by research fora, symposia and discussion groups at various conferences, as well as several special issues of research journals, most recently the *Mathematics Education Research Journal* (Mitchelmore & White, 2007).

One of the approaches to research on abstraction presented on these occasions is *abstraction in context,* or AiC (Hershkowitz, Schwarz & Dreyfus, 2001). This approach considers abstraction as a process of emergence of knowledge constructs that are new to the learner. In order to describe such processes at a fine-grained level, abstraction in context makes use of a model, the RBC model, which is based on three epistemic actions to be described below. The RBC model has been used for this purpose by different research teams with students of different ages learning about different mathematical topics (including square roots, algebra, probability, rate of change, function transformations, and dynamical systems), in a variety of social and learning contexts (see e.g., Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007, and references therein).

Unsurprisingly, students' emerging knowledge constructs may be less complete or less correct than required in the particular situation, and the model has been used to describe the emergence of such *partially correct constructs*, or PaCCs (Ron, Dreyfus & Hershkowitz, 2006). The aim of this paper is to show the analytic power of the RBC model by demonstrating that, and how, it can be used for identifying PaCCs.

Abstraction in Context

Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular, and this has led his collaborators to the idea of "vertical mathematization" (Treffers & Goffree, 1985). Vertical mathematization points to a process of constructing by learners that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means. This process interweaves previous constructs and leads to a new construct.

AiC adopts this view and defines abstraction as a process of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. The genesis of an abstraction passes through a three stage process, which includes the arising of the need for a new construct, the emergence of the new construct, and its consolidation. The need may arise from the design of a learning activity, from the student's interest in the topic or problem under consideration, or from a combination of both; without such a need, however, no process of abstraction will be initiated.

We note that this view of abstraction follows van Oers (2001) in negating the role of decontextualisation in abstraction, and embraces Davydov's dialectic approach (1990) in that it proceeds from an initial unrefined first form to a final coherent construct in a dialectic two way relationship between the concrete and the abstract (see Hershkowitz et al., 2001; Ozmantar & Monaghan, 2007).

Furthermore, we found that activity theory (Leont'ev, 1981) proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, we follow Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy but allows avoiding a number of problems presented by constructivism.

According to activity theory, outcomes of previous activities naturally turn to artefacts in further ones, a feature which is crucial to trace the genesis and the development of abstraction throughout a succession of activities. The kinds of actions that are relevant to abstraction are *epistemic actions* – actions that pertain to the knowing of the participants and that are observable by participants and researchers. Pontecorvo & Girardet (1993) have used this term to describe how children developed their knowledge on a historical issue during a discussion. The observability is crucial since other participants (teacher or peers) may challenge, share or construct on what is made public.

The RBC model

For the above reasons, Hershkowitz et al. (2001) have chosen to use epistemic actions in order to model the central second stage of the process of abstraction. The three epistemic actions they have found relevant and useful for their purposes are Recognizing (R), Building with (B) and Constructing (C). *Recognizing* takes place when the learner recognizes that a specific previous knowledge construct is relevant to the problem he or she is dealing with. *Building with* is an action comprising the combination of recognized constructs, in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. *Constructing* consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner either through verbalization or through action. In the case of action, the learner may but need not be fully aware of the new construct. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to the third stage of the genesis of an abstraction, consolidation. Examples for the three epistemic actions will be given below, in the subsection entitled "Data and Interpretation".

This description of the model raises the question what it is that is being constructed, what constructs the researchers focus on. In some cases, the answer to this question is rather immediate and obvious. In other cases, it requires an in depth analysis of both, the design of the curriculum and the learners' actions. We usually begin by an a priori analysis of the instructional design (in which the designers may participate), which aims at identifying the intended constructs. These constructs are thus not absolute but relative to an instructional activity, a learning unit, or a curriculum. They can be mathematical concepts, methods, strategies, and so on. We call them mathematical *principles*. The RBC analysis of the transcripts of the learning events then focuses mainly on these principles, with the researcher's task being to identify specific learner actions as R- or B-actions with these principles and to identify sequences of R- and B-actions as C-actions of these principles. This analysis may lead to a modification of the list of principles, for example by including alternative constructs made by the learners.

The RBC model constitutes a methodological tool used for realizing the ideas of abstraction in context. In this sense, it has a somewhat technical nature that serves to identify learner actions at the micro-level. On the other hand, the model also has a definite theoretical significance; Hershkowitz (2007) has discussed the theoretical aspects of the model, its tool aspects, and the relationships between them.

Partially Correct Constructs

Ron, Dreyfus and Hershkowitz (2006) have studied cases where students' incorrect answers overshadow meaningful knowledge they have constructed, or cases in which students' correct answers hide knowledge gaps. Both cases may indicate knowledge constructs that are partly but not fully appropriate to the mathematical problem situation at hand. Thus, students' constructs that only partially fit the mathematical principles underlying the learning context have been called *Partially Correct Constructs*, or PaCCs. We stress that PaCCs refer to the principles and thus are relative to the problem situation, the activity, and the curriculum at hand, as well as to the researchers' decision to focus on some knowledge aspects rather than others. PaCCs can be used as tools for interpreting situations in which some answers or actions of a student are inconsistent with others.

Consider, for example, the task in Figure 1. It is expected (see e.g., Nilsson, 2007) that many students predict Ruti to win in question 1b, basing their analysis on a 21 event sample space. In a sample of 120 grade 8 (age 13) students, it turned out that 57 students indeed answered thus. However, almost as many students (35) predicted Ruti to win in spite of answering in 1c that there are 36 possible outcomes. The answers "Ruti wins" and "the sample space has 36 events" appear to be inconsistent.

Such inconsistencies call for explanations. The RBC model for abstraction in context allows the researcher to follow a student's knowledge constructing process in order to identify the student's constructs with respect to the principles underlying the learning context. Thus, the micro analytic nature of the RBC analysis can help us answer the question, which student constructs are partially correct and how they led to the inconsistent answers.

Identification of Partially Correct Constructs: A Case Study

In this section, we will use the RBC model in order to follow the knowledge construction process of one grade 8 student, Roni, working with his peer, Yam, on the task in Figure 1.

1a Yossi and Ruti roll two white dice. They decide that Ruti wins if the numbers of points on the two dice are equal, and Yossi wins if the numbers are different.
Do you think that the game is fair? Explain!

1b The rule of the game is changed. Yossi wins if the dice show <u>consecutive</u> numbers. 2 and 3 are consecutive numbers, or 23, 24 and 25 are consecutive numbers.
Do you think the game is fair?

1ci How many possible outcomes are there when rolling two dice?
An outcome of rolling two dice is, for example, .

1cii Reconsider your answers to tasks 1a and 1b: Are the games fair?

1d If Yossi and Ruti play with one red die and one white die, does this change the answers to 1a, 1b and 1c?

Figure 1. A probability task (translated from Hebrew).

Methodological Considerations

Because of space limitations, we keep this subsection short.

A ten lesson elementary probability unit was designed for grade 8 (age 13) and experimentally taught to six student pairs in laboratory conditions and to seven classes. In each of these classes, one or two focus groups were video-taped throughout the unit, Roni and Yam among them. We refer the reader to Hershkowitz et al. (2007) for more details about the choice of topics and the design of the unit. Here we only stress that the activity based on the task in Figure 1 was the first activity in which the students had an opportunity to deal with a two dimensional sample space.

Principles

We recall that the principles are relative to the design, in our case to the design of the task in Figure 1. Our analysis led to four principles needed to solve the task, namely simple event, sample space, compound event and probability value. While these principles may seem natural and almost trivial to the reader, they are not operational for the researcher who wants to trace the construction of knowledge. For this purpose, we need to decompose the principles into elements that are defined operationally so that we can tell from students' actions whether or not they are using these elements. A full treatment of the principles would need more space than available here; a somewhat partial version follows.

The *simple event* principle: A simple event in two dimensional sample space is an ordered pair of one dimensional simple events, one for each dimension. The simple event principle thus has two elements *pair* and *order*, which are operationally defined as follows: We shall say that students have constructed the pair element when they relate to the occurrence of the two one dimensional simple events as to a single event. We shall say that students have constructed the order element if they relate to pairs like (,Δ) and (Δ,) as to two distinct events. The order element can be constructed only if the pair element has been constructed.

The *sample space* principle has three elements, one of which is *systematic counting*; another one is the *product* element: The number of events in a two dimensional sample space is the product of the numbers of the events in the two relevant one dimensional sample spaces. We shall say that students have constructed this element if they calculate the number of all the events in the sample space by the appropriate multiplication.

The *compound event* principle has two elements, one of which it *systematic counting*.

The *probability value* principle has a single element, *ratio*. We shall say that students have constructed the ratio element if they calculate the probability value as the ratio between the number of simple events in the appropriate compound event and the number of simple events in the sample space.

The principles are connected as follows: the probability value principle contains the compound event and the sample space principles as elements, while the compound event and the sample space principles each contain the simple event principle as an element. Thus each principle in fact contains up to two more elements than enumerated above.

Data and Interpretation

Roni is one of the 35 students in the above sample who correctly answered that there are 36 possible outcomes when rolling two dice. On the other hand he claimed, regarding questions 1b and 1cii (Figure 1), that Ruti has a better chance than Yossi to win the game because she wins with 6 possible outcomes, while Yossi wins only with 5 possible outcomes.

Already while working on question 1a, Roni declared

Roni	7	Each number can come out five [times], true? In order for him to win [with] two: five more; three: five more; four: five more; five: five more; six: five more. Multiply six by five. And six times, six possibilities for Ruti.
Roni	9	One with one, two with two, three with three, four with four, five with five, six with six. That's six out of 36. So let's write it.
Roni	10	[Dictating to Yam] The game is not fair because out of 36 possible outcomes, 30 times Yossi will win and only six times Ruti will win. And in brackets write "on average".

In the interpretation of these (and the following) data, we will rely on the three epistemic actions of the RBC model, recognizing, building with, and constructing. In his answers to question 1a Roni shows evidence for having constructed several of the knowledge elements listed above, in the context of the specific task he is working on, including the pair element: In R9 he explicitly refers a pair of one dimensional outcomes as a simple event to be counted. In his further work, Roni consistently recognizes the pair element and builds with it. In R7 we also have evidence of having constructed the systematic counting elements both, for the sample space size and for the relevant events that are included in the compound event. We note that Roni found the number of possible outcomes by calculating a product (R7). In R9 and in the written answer he refers to the probability to win the game as the ratio between the number of simple events of the compound event and the number of the events in the sample space. Roni had constructed the ratio element at an earlier stage, when dealing with one dimensional sample spaces. Now he recognizes this knowledge element as relevant for the new context of two dimensional sample space, and builds his calculations with it. However, so far we have no evidence about Roni constructing the order element.

Roni and Yam proceed to question 1b. Yam speaks first, spelling out five pairs of consecutive numbers. Roni soon dictates the common answer.

Roni	29	Write: The game is not fair because out of …
Roni	31	… eleven possible outcomes, six times …
Roni	33	Six times …, write in parentheses “on average”, Ruti will win, versus, versus five times, that Yossi will win.

Roni did not use the order element. While Yam was the first to count only five of the ten consecutive pairs, Roni did not correct him, and dictated their common answer. We further note that, for the second time, Roni stressed the meaning of probability as an average behaviour, thus showing a deep understanding of the probability concept.

The students turn to question 1c. Despite the fact that Roni has talked about 36 possible outcomes earlier, he now stops to think again, but when Yam starts enumerating events, Roni stops him and declares

Roni	38	Every one has six. Thirty six possibilities. Every number six times.

We can see that Roni has constructed the product element. He correctly determines the number of possible outcomes without referring to each of them.

The next question asks them to reconsider their decisions whether the games are fair. Roni immediately says that the games are not fair. He explains why to the teacher who happens to approach their desk and asks for clarification. Roni’s explanation is a repetition of his earlier claim. When approaching question 1d, Roni soon stops to reflect.

Roni	69	The same.
Yam	70	The same.
Roni	71	The same but here there is a red die. Here [points to the white pair of dice] if you get five, two, it is impossible to then get two, five. This is considered the same thing. But here [points to the red/white pair of dice] it is possible. If you get, say, five red and two white, maybe it’s possible five white and two red.

The written answer that Roni dictates indicates that he may now have constructed the order element in one context:

Roni	79	There will be a different answer because it can be 5 on the white die and 2 on the red one; and the opposite is considered a different answer.

However, even now that Roni is aware of the existence of pairs of symmetric events, at least in the context of dice with different colours, he does not connect this awareness to the game of Yossi and Ruti who play with a pair of white dice, and does not change his mind regarding the fairness of the game in question 1b.

We note in passing that Roni’s explanation in 71 and 79 clearly shows that his count of 5 consecutive pairs is not due to the fact that he considered only increasing pairs like (2,3) as consecutive and not decreasing ones like (3,2); rather his count of 5 is due to explicit identification of symmetric pairs in the case of dice of the same colour. We have analogous evidence about other students’ reasons for counting 5 rather than 10 consecutive pairs; and we have no evidence of any students having discounted decreasing pairs.

We further note that Roni’s construction of the order element is an example for the role of context in processes of abstraction. Here, the context is mathematical, but social context or learning context such as technological tools have been shown to be equally crucial to processes of abstraction (see e.g., Hershkowitz et al., 2007).

Analysis and Discussion

We defined PaCCs as constructs that only partially match the relevant underlying mathematical principles and their elements. Thus constructs are considered to be partial with respect to the mathematical principles that underlie a specific learning unit, design, or activity. A mathematical principle is then considered a PaCC if only some of its elements have been constructed and are recognized as relevant when appropriate, whereas others may be lacking or may themselves be PaCCs. The analysis leading to the identification of PaCCs now consists of tracing every knowledge element from the time it was first constructed through recognizing it as relevant for other contexts, and through building with it further. The RBC analysis is thus used as a tracer for identifying cases of partial match between a student’s construct and the corresponding mathematical principle.

In order to carry out the analysis in the specific case at hand, we consider Roni's constructions concerning the four principles underlying the current stage of the learning design. Concerning the simple event principle, the data show that at an early stage of his learning process Roni constructed the pair element without the order element. While he worked on question 1d, he constructed the order element for the context of different coloured dice. However, even then, he claimed that when playing with two white dice, one should not consider (2,5) and (5,2) as different outcomes. We thus consider Roni's construct for simple event as a PaCC since the order element remained unconnected in the context of question 1b.

Since there is no space here for a more detailed analysis, we summarize the discussion of the other three principles as follows: the other three principles all inherit the property of being PaCCs from the simple event principle. Since simple event is an element of sample space and simple element is a PaCC, sample space is a PaCC as well; the same argument holds for compound event, and a similar but less direct argument holds for the probability value principle.

As one can see, the only knowledge element that was not constructed by Roni until a rather late stage is the order element. This element remained disconnected until the end of the work on Task 1. The order element is an element of the simple event principle. However, not only the simple event principle is considered as a PaCC, but also all those principles that are built on it are considered as PaCCs because they inherit the PaCC property from it. An example of the expression of the lack of the order element in a higher principle is Roni's determination of five consecutive pairs in the compound event "Yossi wins".

Conclusion

By following students' knowledge construction processes at a micro-level, the RBC model for abstraction in context can be used to explain students' inconsistent answers via the notion of partially correct constructs. Roni's case has been presented above. Similar analyses of other students' constructs for the order element as well as of Roni's constructs for more sophisticated elements of probabilistic knowledge, specifically for the area model, have met with analogous success in explaining inconsistent answers.

These results have been achieved by using the epistemic actions of the RBC model as tracers. The use of the epistemic actions as tracers is thus an efficient tool for identifying the nature of PaCCs. Using the epistemic actions as tracers requires using them in the order CRB: Once there is evidence of construction, the construct is traced to see whether the student recognizes and builds with it in later situations, where it is relevant. The student's failure to recognize and build with the construct where it is relevant indicates a PaCC. In the specific example described above, Roni builds with the product principle without being bothered whether he counts ordered or non-ordered pairs. He did construct the order principle when working on question 1d, but only for a narrow context in which one can distinguish between the two dice and hence assign an order to the pair of dice. In the game of question 1b, however, the dice are not distinguishable. Roni does not recognize the order principle as relevant to that context and, therefore, does not experience any need to reconsider the fairness of the games, even when asked to do so.

The RBC model of abstraction in context has thus been proved its power as an analytic tool that allows the researcher not only to identify a student's PaCCs but also to determine the precise nature of these PaCCs in terms of knowledge elements that have been constructed and others that have not.

References

Davydov, V. V. (1990). *Soviet studies in mathematics education: Vol. 2. Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Kilpatrick, Ed., & J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published in 1972).

Giest, H. (2005). Zum Verhältnis von Konstruktivismus und Tätigkeitsansatz in der Pädagogik. In F. Radis, M.-L. Braunsteiner, & K. Klement (Eds.), *Badener VorDrucke* (pp. 43-64). Baden A.: Kompetenzzentrum für Forschung und Entwicklung (Schriftenreihe zur Bildungsforschung - Band 3).

Hershkowitz, R. (2007). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. B. Schwarz & T. Dreyfus (Eds.), *Guided construction of knowledge in classrooms* (Proceedings of an Israel Science Foundation workshop, pp. 39-49). Jerusalem, Israel: The authors.

Hershkowitz, R., Hadas, N., Dreyfus, T., & Schwarz, B. B. (2007). Abstracting processes, from individuals' constructing of knowledge to a group's 'shared knowledge'. *Mathematics Education Research Journal, 19*(2), 41-68.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32* (2), 195-222.

Leont'ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 37-71). Armonk, NY: Sharpe.

Mitchelmore, M., & White, P. (2007). Abstraction in mathematics learning. *Mathematics Education Research Journal, 19*(2), 1-9.

Nilsson, P. (2007). Different ways in which students handle chance encounters in the explorative settings of a dice game. *Educational Studies in Mathematics*, *66*, 273-292.

Ozmantar, M.F., & Monaghan, J. (2007). A dialectical approach to the formation of mathematical abstractions. *Mathematics Education Research Journal, 19*(2), 89-112.

Pontecorvo, C., & Girardet, H. (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction, 11*(3&4), 365-395.

Ron, G., Dreyfus, T., & Hershkowitz, R. (2006). Partial constructs for the probability area model. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 449-456). Prague, Czech Republic: PME.

Treffers, A., & Goffree, F. (1985). Rational analysis of realistic mathematics education – The Wiskobas program. In L. Streefland (Ed.), *Proceedings of the 9th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 97-121). Utrecht, the Netherlands: PME.

Van Oers, B. (2001). Contextualisation for abstraction. *Cognitive Science Quarterly, 1*, 279-305.