Students' Conceptual Understanding of Equivalent Fractions

Monica Wong
University of Sydney
<monica.wong@optusnet.com.au>

David Evans
University of Sydney
<d.evans@edfac.usyd.edu.au>

This paper investigates students' conceptual understanding of equivalent fractions by examining their responses to questions using symbolic and pictorial representations. Two hundred and thirteen students in Years 3 to 5 from three Sydney primary schools were administered a general mathematics achievement test and a fraction assessment. Five questions from this fraction assessment instrument were analysed. The different types of knowledge used to answer each question were examined and common misconceptions identified. The responses of students with limited general mathematics achievement were compared to those of their more competent peers. The differences that emerged between the two groups in their conceptual understanding of equivalent fractions, were highlighted.

The development of conceptual understanding involves seeing the connections between concepts and procedures, and being able to apply mathematical principles in a variety of contexts. It is a central focus of the NSW Mathematics curriculum (Board of Studies NSW (BOS NSW), 2002). Considering the difficulties experienced by students in mastering equivalent fractions and the many misconceptions they hold (e.g., Gould, 2005a, 2005b; National Research Council (NRC), 2001; Pearn, Stephens, & Lewis, 2003), identifying the nature of the differences in conceptual understanding between students of varying levels of general mathematical proficiency provides a mechanism to inform the teaching of this particular concept (NRC, 2001).

As part of a larger study which examined students' conceptual understanding of equivalent fractions, an Assessment of Fraction Understanding (AFU) instrument was developed. The pencil and paper test contained 34 questions that were used to measure students' conceptual understanding, their ability to solve routine problems and to adapt their understanding to non-routine problems (NRC, 2001; Shannon, 1999). Three schools participated in this phase of the study. All students were administered the AFU instrument and some students also participated in semi-structured interviews.

This paper focuses specifically on five fraction questions from the AFU and their diagnostic potential in identifying students' misconceptions. Comparisons between the responses of students with naïve and with more advanced mathematical understanding assist in defining the progressive learning sequences followed by students to master and understand equivalent fractions.

Theoretical Perspective

Systems of Knowledge

Mathematics is a reasoning activity that involves observing, representing and investigating relationships in the social and physical world, or between mathematical concepts themselves (BOS NSW, 2002). A mathematical concept is not a single isolated idea but one idea in a structured system of knowledge or schemata (Anderson, 2000; Lesh, Landau, & Hamilton, 1983). Information-processing models of cognitive development suggest that within these structured systems of knowledge, information stored in memory can be categorised into declarative and procedural knowledge (Anderson, 2000).

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

Declarative knowledge is knowledge of specific facts and ideas (Anderson, 2000). Mathematical definitions of procedural knowledge assume a foundation of declarative knowledge: "a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols" (Hiebert & Lefevre, 1986, p. 7). Procedural knowledge also incorporates the awareness of how to approach a task and its related steps or algorithms (Anderson, 2000).

Conceptual understanding in mathematics develops when students "see the connections among concepts and procedures and can give arguments to explain why some facts are consequences of others" (NRC, 2001, p. 119). Facts are no longer isolated but become organised in coherent structures based on relationships, generalisations and patterns, Conceptual understanding has also been described as "conceptual knowledge" (Anderson, 2000; Rittle-Johnson, Siegler, & Alibali, 2001) and "relational understanding" (Skemp, 1986). Rittle-Johnson et al. (2001) found that developing students' procedural knowledge had positive effects on their conceptual understanding, and conceptual understanding was a prerequisite for the students' ability to generate and select appropriate procedures.

Thus, conceptual understanding is intertwined with procedural knowledge. This makes the isolated study of either difficult, requiring more than the determination of the correctness/incorrectness of a student's answer. It requires further investigation into the response, which can provide valuable insight into the thinking (Gould, 2005a; 2005b).

Fraction Knowledge

A *common fraction* (fraction) is often described as the ratio or quotient of two whole numbers, *a* and *b*, expressed in symbolic form $\frac{a}{b}$, where *b* is not zero (BOS NSW, 2002). It is a symbol that has meaning and can be interpreted and manipulated. The fraction schemata includes five interconnected, yet distinct interpretations (Lamon, 2001), as shown in Table 1. Using these interpretations, one can explore the various characteristics and manipulations of fractions (such as proper and improper fractions, mixed numerals, fraction equivalence, comparison, addition, multiplication and division). The concept of fractions is also linked to other mathematical concepts such as geometry, number-lines, and whole number multiplication and division.

Table 1

Different Fraction Interpretations for the fraction $\frac{3}{4}$

Interpretations	Example
Part/whole	3 out of 4 equal parts of a whole or set of objects or collection
Measure	$\frac{3}{4}$ means a distance of 3 ($\frac{1}{4}$ units) from 0 on the number line
Operator	$\frac{3}{4}$ of something, stretching or shrinking
Quotient	3 divided by 4, $\frac{3}{4}$ is the amount each person receives
Ratio	3 parts cement to 4 parts sand

Fraction concepts can be explained by teachers and students using a combination of external representations such as written symbols, spoken language, concrete materials, pictures, and real world examples (Lesh et al., 1983).

Conceptual Understanding of Fraction Equivalence

Fraction equivalence is one concept within the extensive fraction schemata. Equivalence implies similar worth. Thus two common fractions are considered equivalent when they have the same value (BOS NSW, 2002; Skemp, 1986). A fraction represents a number with an infinite number of names. Listing some of these names makes it apparent that each individual fraction is part of an "equivalence set". For example, the equivalence set for the fraction $\frac{1}{2}$ can be represented as $[\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, ...]$. Implicit in the concept of equivalence is the knowledge that each fraction in the set is interchangeable with the others.

Conceptual understanding of equivalent fractions involves more than remembering a fact or applying a procedure. It is based on an intricate relationship between declarative and procedural knowledge; between fraction interpretation and representation. Students should able to: (a) make connections between fraction models by understanding the sameness and distinctness within these interpretations (Lesh et al., 1983; NRC, 2001); (b) make connections between the different representations (Lesh et al., 1983), and (c) show that a fraction represents a number with many names. The present study examines a small portion of the large body of knowledge associated with fractions.

Figure 1 depicts the scope of the questions used to identify students' conceptual understanding of equivalent fractions. At the lowest level, knowledge is declarative and procedural, loosely linked to specific examples of equivalent fractions (NRC, 2001) and not generalised across representations or interpretations. As students develop understanding, their knowledge becomes generalised and applied more broadly.

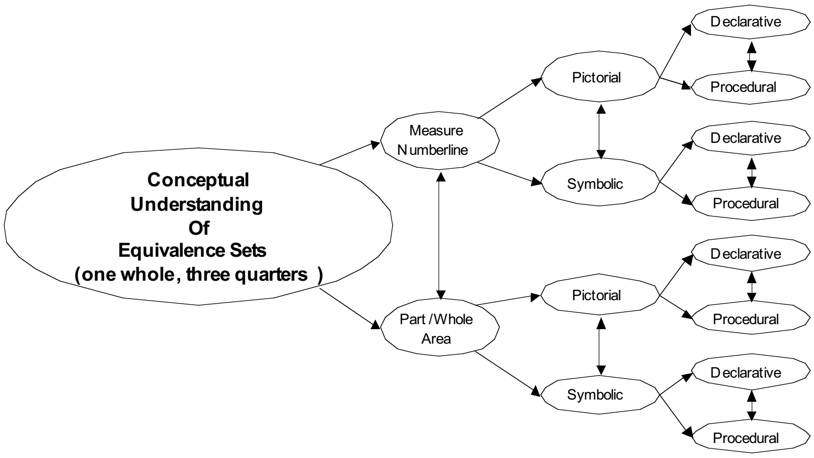
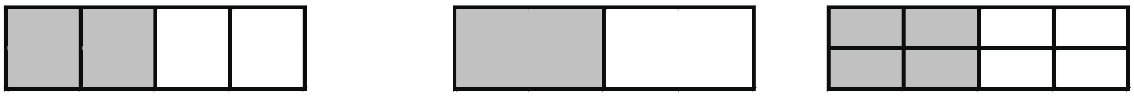


Figure 1. Model used to develop equivalent fraction questions (Adapted from Lamon (2001, p.151)).

In this study, students were presented with tasks that aimed to elucidate their level of thinking. The demands of the tasks were restricted to identifying symbolic and pictorial representations and representing fractions using part/whole area and measure models. They incorporated "skill" questions that required the recall of a practised routine or procedure, and "conceptual" questions that required students to apply their knowledge and explain their actions (Shannon, 1999).

Tasks that incorporate pictorial representations with visual distractors provide one method of measuring students' conceptual understanding of equivalent fractions. Such tasks have been found to highlight the unstable nature of a student's fraction knowledge

(Ni, 2001; Niemi, 1996). Pictorial representations of part/whole area and measure models can be described as "simple representations" when the total number of equal parts in the shape matches the fraction denominator. They allow students to count the parts (see Figure 2a). The shaded part is associated with the numerator and the entire shape is associated with the denominator. Equivalent pictorial representations are visually challenging. They occur when the number of equal parts of the whole is a multiplicative factor less or greater than the denominator (Niemi, 1996), as shown in Figures 2b and 2c. The areas of the whole and shaded part never change, but the number of equal parts into which the whole is divided can alter dramatically. Thus different fraction names can be offered for the shaded area and an equivalence set identified. Simple and equivalent representations for a measure model appear in Figure 3.



(a) simple (b) equivalent - 2 equal parts (c) 8 equal parts

Figure 2. Part/whole area model simple and equivalent representations for two quarters.

(a) simple (b) equivalent - 8 equal parts

Figure 3. Measure model simple and equivalent representations for two quarters.

Equivalent fraction tasks using symbolic notation (see Figure 4) are more cognitively demanding as up to four dimensions need to be simultaneously co-ordinated: the original two-dimensional fraction, $\frac{3}{8}$ and its equivalent, $\frac{12}{32}$ (English & Halford, 1995). Questions that incorporate the interpretation and manipulation of symbolic notation are ideal for identifying the levels of students' conceptual understanding of equivalent fractions. Evaluation of their responses provides an insight into the students' thought patterns, conceptual understanding and procedural knowledge. Teachers who understand how students develop this knowledge, and are able to help them to see the links between various representations are providing the most effective fraction programs for students.

(a) $\frac{3}{8} = \frac{\_\_}{32}$ (b) $\frac{3}{8} = \frac{12}{\_\_}$ (c) $\frac{3}{8} = \frac{\_\_}{\_\_}$ Answer for a and b: $\frac{3}{8} = \frac{12}{32}$

Figure 4. Typical equivalent fraction question and answer employing symbolic representations only.

The purpose of this study was to evaluate students' understanding of equivalent fractions through their responses to questions that incorporated symbolic and pictorial representations, and required them to identify measure and part/whole interpretations. Firstly, the types of knowledge used by students to answer these questions were investigated. Secondly, responses by students of varying general mathematical achievement were compared to examine the differences evident in their developing mastery of equivalent fractions.

Methodology

Participants

Two hundred and thirteen students from Years 3 to 5 from three Sydney primary schools participated in the study. Their details appear in Table 2.

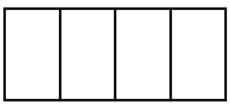
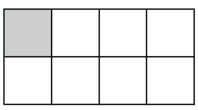
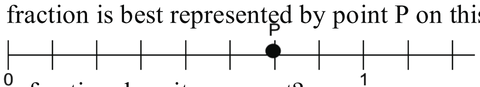
Table 2
Participant Details

Grade level	Sample size (n)	Age (years)		Gender	
		Range	Avg.	% Boys	% Girls
3	64	8.15-10.21	8.84	48.4	51.6
4	80	7.97-11.08	9.84	50.0	50.0
5	69	10.02-12.75	10.81	37.7	62.3

Instruments

The *Progressive Achievement Tests in Mathematics* (PATMaths) was used to measure students' general mathematics achievement (Australian Council for Educational Research [ACER], 2005). As recommended by ACER, different tests were used for grades 3 to 5. All tests were norm referenced and scores calibrated on a common scale. The questions for the Assessment of Fraction Understanding (AFU) were derived and adapted from various assessment instruments including the Trends in Mathematics and Science Study, the North Carolina Testing Program, the California Standards Test and the Success in Numeracy Education program (Catholic Education Office, 2005). The questions analysed in this paper related to the fraction "one whole" and "three quarters" and appear in Table 3, along with the representation mapping used for each question.

Table 3
Questions Analysed

Fraction	Symbolic to Pictorial	Symbolic to Symbolic
One whole	14. Shade in $\frac{2}{2}$ of the shape below?  Can you think of another name for the fraction shaded?	29. Circle the fractions that are equal to 1? $\frac{8}{8}$ $1\frac{1}{100}$ $\frac{1}{1}$ $\frac{9}{10}$ $\frac{4}{4}$ $1\frac{1}{8}$ $\frac{7}{8}$ $\frac{10}{9}$ $\frac{9}{8}$ How did you work this out?
Three quarters	13. In the figure, how many small squares need to be shaded so that $\frac{3}{4}$ of the small squares are shaded? 	28 (b). $\frac{6}{8} = \frac{\square}{\square}$
	Pictorial to Symbolic	
Three quarters	18. What fraction is best represented by point P on this number line? _____  What other fraction does it represent?	

The questions were linked to the *Mathematics K-6 Syllabus*, as shown in Table 4. Stage 2 (NS2.4) knowledge and skills are generally taught in years 3 to 4, whereas Stage 3 (NS3.4) skills are taught in years 5 to 6. All questions were open-ended, which allowed for students' understanding to be examined more effectively. Part/whole area questions used an equivalent area representation, and the measure question used a simple number-line representation. Questions 29 and 14 examined the concept of one whole, whereas questions 13, 28b, and 18 examined three quarters. Question 18 further illuminated the sophistication of the students' connections between measure and part/whole interpretations.

Table 4
Mathematics K-6 Syllabus Reference (BOS NSW, 2002)

Question	Syllabus Reference (knowledge and skills)	
14, 29	NS2.4 (1)	Renaming $\frac{2}{2}, \frac{4}{4}, \frac{8}{8}$ as 1
13	NS2.4 (2)	Finding equivalence between halves, quarters and eighths using concrete materials and diagrams, by re-dividing the unit
28(b)	NS3.4 (2)	Developing a mental strategy for finding equivalent fractions, e.g., multiply/divide the numerator and the denominator by the same number
18	NS2.4 (2)	Placing halves, quarters and eighths on a number line between 0 and 1 to further develop equivalence

Procedure

All participants were tested during term three, 2006, over two consecutive days. The PATMaths test was administered on the first day and the AFU the following day. Both tests were administered following standardised protocols. Each pencil and paper test was of 45 minutes duration. Calculators were not permitted. Participants were asked to show all working for the AFU in their test booklet.

Results

Most Australian states and territories identify students "at risk" as the lowest achieving 20 percent of students (Doig, McCrae, & Rowe, 2003). Participants with limited *general mathematics achievement* (GMA) are identified as those students who score below the 20th percentile on their particular PATMaths test, when compared with the norming data (ACER, 2006). Students scoring in the middle 60% are considered to be developing mathematical knowledge at an appropriate level, whilst the upper most 20% are identified as more competent. Participants were categorised into achievement levels (see Table 5).

Table 5
Student Achievement

	General Mathematics Achievement (GMA)							
	Limited (N = 28)		Avg. (N = 152)		High (N = 33)		Total (N = 213)	
Grade	*n*	% of Grade	*n*	% of Grade	*n*	% of Grade	*n*	% of Total
3	6	9.4	45	70.3	13	20.3	64	30.0
4	8	10.0	61	76.3	11	13.8	80	37.6
5	14	20.3	46	66.7	9	13.0	69	32.4
Total	28	13.1	152	71.4	33	15.5	213	100.0

A Rasch analysis was conducted to determine the difficulty of each question used in the AFU. The relative difficulty of each item and other associated Rasch statistics are shown in Table 6. The 'fit residual' statistic confirms whether the item is over or under-discriminating in comparison to the theoretical dichotomous Rasch model (which has an acceptable fit statistic between -2 and 2) (Bond & Fox, 2001). The chi-square probability statistic verifies whether there is a statistically significant difference between the theoretical and observed item discrimination for each question (RUMM Laboratory, 2004). There were no significant deviations from the theoretical Rasch model for any of the five questions analysed in this study.

Table 6
Item Difficulty

Question	Difficulty	SE	Fit Residual	Chi-square	*p*
29. Circle fractions equal to 1	-0.691	0.150	-1.851	9.359	0.052
14. Shade in 2/2 of the shape	0.061	0.141	0.533	5.484	0.241
13. Shading 3/4 of small squares	0.211	0.141	-1.145	6.917	0.140
28 (b) 6/8 =	0.611	0.145	-0.564	2.729	0.604
18. Fraction represented on a number-line	1.821	0.157	-0.291	4.875	0.300

The easiest questions (i.e., 29 and 14) required students to identify one whole. Students were more able to identify three quarters of an equivalent area model than 1) to determine an equivalent fraction for three quarters using only symbolic representation or 2) to identify a fraction using a measure model.

Further question analysis identified the knowledge structures participants employed to solve these equivalent fraction problems. Commencing with the easiest question (29), Table 7 shows the percentages of students who answered the question correctly and incorrectly. Eighty percent of students who answered the question justified their response by stating that the top number and bottom number were the same. Participants explained their thinking by using procedural knowledge, which does not exclude conceptual understanding. The participants who provided an incorrect response provided no observable pattern of reasoning. From the incorrect responses given, many students selected fractions that contained the number 1 as part of the fraction (either 1/1 or mixed numerals containing the whole number 1).

Table 7
Responses to Question 29: Circle the Fractions that are Equal to 1 (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
Answer selected	*n*	%	*n*	%
CORRECT:1/1, 4/4 and 8/8 selected	10	35.7	25	75.8
1/1 only	5	17.9	3	9.1
Two or more of the following selected: 1/1, 1 1/8, 1 1/00	7	25.0	1	3.0
Other	4	14.3	2	6.0
No response	2	7.1	2	6.0

The application of students' knowledge in linking symbolic to pictorial representations of "one whole" was tested in question 14 using an equivalent pictorial representation. Responses are tabulated in Table 8. Nearly all the participants who were able to answer the question correctly were also able to give another name for the fraction shaded. Only 10.7% (n = 3) of participants with limited GMA and 54.5% (n = 18) of participants with high

GMA were able to answer questions 29 and 14 correctly. Thus, these participants were able to show greater conceptual understanding as they applied their symbolic understanding of one whole to an equivalent pictorial representation. For those participants who answered the question incorrectly (shading 2 small squares), approximately half wrote “1/2” for the fraction shaded.

Table 8
Responses to Question 14: Shade in 2/2 of the Shape (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
Number of small squares shaded	*n*	%	*n*	%
CORRECT: 4	5	17.9	22	66.7
Participants are able think of another name for the fraction shaded	4	14.3	20	60.1
2	22	78.6	10	30.3
Participants gave response 1/2 for fraction shaded	9	32.1	5	15.2
Other or Missing	1	3.6	1	3.0

Although 3/4 is a commonly presented fraction, low GMA participants had difficulty representing the fraction using an equivalent part/whole area diagram. Responses from all participants for question 13 are shown in Table 9. Their most common incorrect response was to shade three small squares. Six of these participants also shaded 2 squares in question 14, suggesting they used the value of the numerator in both questions to determine the number of squares to shade.

Table 9
Responses to Question 13: How Many Small Squares Need to be Shaded (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
Number of small squared shaded	*n*	%	*n*	%
CORRECT: 6	4	14.3	23	69.7
2	5	17.9	1	3.0
3	12	42.9	8	24.2
1, 4, 5	4	14.3	0	0.0
Missing	2	7.1	1	3.0

Question 28b presented a symbolic to symbolic equivalent fraction question and participant responses are shown in Table 10. This question can be solved procedurally by multiplying the top and bottom by the same number. Some participants gave either the response 4/6 or 8/10, indicating that they may have separated the fraction into two components, with the bottom number being two greater than the top one. An equivalent fraction was then constructed with a similar pattern. Three limited GMA students answered questions 13 and question 28b correctly. For the high GMA group, 14 participants answered questions 13 and 28b correctly. Only 2 limited GMA students answered all four questions 29, 14, 13, and 28 correctly compared to 11 from the high GMA group.

The number of participants who were able to identify point P on the number line is shown in Table 11. Only 33.3% (n = 11) of the high GMA group answered the question correctly. Only seven of these participants were able to list another name for the fraction. These seven participants answered all three “3/4” questions and question 29 (identifying symbolic representations of one whole) correctly. Only four of these seven participants answered all five questions correctly. It was these four participants who showed the greatest conceptual understanding of equivalent fractions, as they were not only able to link symbolic and pictorial representations but also offer another name for a specific fraction and apply their knowledge across different fraction representations consistently. No participants in the low GMA group were able to answer all questions correctly. They did not apply their knowledge consistently across representations and were unable to transfer their knowledge to the measure interpretation.

Table 10
Responses to Question 28b: 6/8 = __/__ (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
	n	%	*n*	%
CORRECT equivalent fraction given	7	25.0	18	54.5
4/6 or 8/10	2	7.1	4	12.1
Other	8	26.6	7	21.2
Missing	11	39.3	4	12.1

Table 11
Responses to Question 18: Identify point P on the number line (n=61)

	Limited GMA (N = 28)		High GMA (N = 33)	
	n	%	*n*	%
CORRECT: 6/8	1	3.6	8	24.2
CORRECT: 3/4	1	3.6	3	9.1
6/10	0	0.0	3	9.1
6	3	10.7	2	6.1
Other	15	53.6	9	27.2
Missing	8	26.8	8	24.2

Discussion

Students' conceptual understanding of equivalent fractions was examined in this study through their responses to mathematical problems that required them to make connections between equivalent pictorial and symbolic representations incorporating measure and part/whole area interpretations.

Students demonstrated the use of procedural knowledge when answering equivalent fraction problems presented in symbolic form. In some instances, whole number reasoning was exhibited in the procedures they used. Many students were unable to represent a symbolic fraction using an equivalent area diagram. Students who successfully linked symbolic and pictorial part/whole area interpretations for one whole and three quarters showed their knowledge was more generalised and were more able to apply their understanding to pictorial representations using a number-line (measure interpretation). However, the difference between the students in the limited and the high general mathematics achievement groups seems to lie not in the errors they made as similar types of errors were observed. Rather, the depth of their procedural and declarative knowledge and the strength of their connections between procedures and concepts varied as shown in the percentage of questions answered correctly and the types of questions answered correctly.

Conceptual understanding and procedural knowledge are delicately intertwined. The analysis of additional questions or the interview data may assist in clarifying students' level of conceptual understanding. It may also corroborate the findings of Siemon, Izard, Breed, and Virgona (2006) who demonstrated that students with developing fraction knowledge were able to perform simple fraction tasks, but were unable to explain or justify their thinking in writing.

References

Anderson, J. R. (2000). *Cognitive psychology and its implications* (5th ed.). New York: Worth Publishers.

Australian Council for Educational Research. (2005). *Progressive achievement tests in mathematics* (3rd ed.). Melbourne: ACER Press.

Board of Studies NSW. (2002). *Mathematics K-6 syllabus*. Sydney: Author.

Bond, T. G., & Fox, C. M. (2001). *Applying the rasch model: Fundamental measurement in the human sciences*. Mahwah, NJ: Erlbaum.

Catholic Education Office. (2005) *Success in numeracy education [SINE] YEARS 5-8: Fraction screening tests and interviews.* Melbourne: Author.

Doig, B., McCrae, B., & Rowe, K. (2003). *A good start to numeracy*. Canberra: Commonwealth Department of Education, Science and Training.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum Associates.

Gould, P. (2005a). Drawing sense out of fractions. In M. Coupland, J. Anderson & T. Spencer (Eds.), *Making mathematics vital* (Proceedings of the 20th Biennial Conference of the Australian Association of Mathematics Teachers, pp. 133-138). Adelaide: AAMT.

Gould, P. (2005b). Year 6 student's methods of comparing the size of fractions. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: research, theory and practice* (Proceedings of the Annual conference of the Mathematics Education Research Group of Australasia, pp. 393-400). Sydney: MERGA.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Lamon, S. J. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 146-165). Reston, VA: The National Council of Teachers of Mathematics.

Lesh, R. A., Landau, M., & Hamilton, E. (1983). Conceptual models and applied mathematical problem-solving research. In R. A. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 263-341). Orlando, FL: Academic Press, Inc.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Education*, *26*, 400-417.

Niemi, D. (1996). *Instuctional influences on content area explanations and representational knowledge: Evidence for the construct validity of measures of principled understanding. CSE technical report 403.* Los Angeles: National Center for Research on Evaluation, Standards, and Student Testing, University of California.

Pearn, C., Stephens, M., & Lewis, G. (2003). Assessing rational number knowledge in the middle years of schooling. In M. Goos & T. Spencer (Eds.), *Mathematics: Making waves* (Proceedings of the 19th Biennial Conference of the Australian Association of Mathematics Teachers, pp. 170-178). Adelaide: AAMT.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*, 346-362.

RUMM Laboratory. (2004). Interpreting RUMM2020: Part I: Dichotomous data: RUMM Laboratory Pty Ltd.

Shannon, A. (1999). *Keeping score*. Washington D.C.: National Academies of Science.

Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). The derivation of a learning assessment framework for multiplicative thinking. In J. Norotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30thannual conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp.113-120). Prague: PME.

Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). London: Penguin Books.