Procedural Complexity and Mathematical Solving Processes in Year 8 Mathematics Textbook Questions

Jill Vincent
University of Melbourne
<jlvinc@unimelb.edu.au>

Kaye Stacey
University of Melbourne
<k.stacey@unimelb.edu.au>

This study examines the procedural complexity and mathematical solving processes required by problems on two topics in seven Year 8 textbooks from four Australian states. The study used definitions from the 1999 TIMSS Video Study. Although variation existed between textbooks, the majority of problems were of low procedural complexity, requiring only the practising of procedures. The general picture was consistent with that painted by the Video Study, with a somewhat stronger emphasis on procedural work.

The 1999 Third International Mathematics and Science Study (TIMSS) Video Study described teaching practices in eighth-grade mathematics and science in the United States and in six countries where students performed well relative to the United States on the TIMSS 1995 assessments. Countries participating in the mathematics component of the TIMSS 1999 Video Study were: Australia, the Czech Republic, Hong Kong SAR, Japan, the Netherlands, Switzerland, and the United States.

Many common features were apparent across the seven countries, for example, teachers in all seven countries talked more than the students, at a ratio of at least 8:1; mathematics teachers in all countries organised the average lesson to include some public whole-class work and some private individual or small-group work; and on average at least 80% of lesson time was spent in solving mathematical problems. Almost 15000 mathematics problems were analysed, with 82% of the problems focusing on number, geometry, and algebra.

There were some features of the 87 randomly selected Australian mathematics lessons that many mathematics educators would find disturbing. Three quarters of the problems presented in the Australian lessons were repetitions of the preceding problems, the highest proportion of the seven countries. The Australian lessons also included the highest proportion of problems of low procedural complexity (77%) and virtually no Australian lessons included verification of results by logical reasoning (Hiebert et al., 2003). This cluster of features of Australian lessons – *low complexity* of problems, which are undertaken with *excessive repetition*, and *absence of mathematical reasoning* in classroom discourse – together constitute what we have termed the "shallow teaching syndrome" (Stacey, 2003).

The Study

This paper presents findings from an early stage of an investigation into the shallow teaching syndrome – whether it is a real pattern or just an artifact of the definitions and procedures of the Video Study, (if real) whether it is indeed undesirable, and (if real) whether it is most evident in "textbook teaching". With this motivation, we set our first goal to compare "textbook teaching" with the findings of the Video Study, asking if the general picture revealed by the Video Study would arise if all lessons followed textbooks

Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia
J. Watson & K. Beswick (Eds), © MERGA Inc. 2007

exactly. This study is also intended to provide insight into the way in which the classifications of problems used for the Video Study operate in practice.

Three Classifications of Problems

Procedural complexity was defined in the Video Study in terms of the number of steps required to solve a problem by a standard method and whether the problem comprised sub-problems. (Details are given in Methodology.) Table 1 shows the average percentage of problems at each level of procedural complexity for Australia, Japan and the Netherlands. Other countries had from 63% to 68% of problems of low complexity.

Table 1
Average Percentage of Problems per Eighth-grade Mathematics Lesson at each Level of Procedural Complexity for Australia, Japan and the Netherlands

	Low	Moderate	High
Australia	77	16	8
Japan	17	45	39
The Netherlands	69	25	6

Note: The percentages do not all sum to 100 because of rounding.

Problems solved in the lessons were also classified according to the mathematical solving processes involved. Three categories were used: *using procedures*, *stating mathematical concepts,* and *making connections* (see Methodology for definitions). The majority of lessons in all countries except Japan were found to have a high proportion of problems per lesson that focused on *using procedures*, with smaller percentages of problems focusing on *stating concepts* or *making connections*. Table 2 gives the average percentage of problems per lesson in these three categories for Australia, Hong Kong SAR and The Netherlands (see Hiebert et al., 2003, p. 99). In addition to Japanese lessons having the highest percentage of problem statements focusing on *making connections* (54%), 39% of lessons contained a proof. Contrasting sharply with the Japanese lessons, virtually none of the lessons from Australia, the Netherlands, and the United States contained instances requiring verification or demonstration by reasoning that a result must be true (a sub-category of *making connections* problems).

Table 2
Average Percentage of Problem Statements per Eighth-grade Mathematics Lesson Focusing on Different Types of Mathematical Solving Processes for Three Countries

	Using procedures	Stating concepts	Making connections
Australia	61	24	15
Hong Kong SAR	84	4	13
The Netherlands	57	18	24

Note: The percentages do not all sum to 100 because of rounding

Where problems were solved publicly, the Video Study compared the implied solving process and the actual solving process. Problems that were intended to engage students in *stating concepts* or *making connections* frequently only exhibited *using procedures* when discussed publicly. In the Australian lessons, for only 8% of problems categorised as *making connections* did the public explanation explicitly draw attention to these connections. Problems were also classified as either exercises or applications (see Methodology for definitions). In the Australian lessons, 45% of problems were applications, compared with 74% for Japan and 34% for the United States.

Characteristics of Textbooks

Textbooks or worksheets were used in at least 90% of the mathematics lessons in all countries (Hiebert et al., 2003). The analysis of textbook questions therefore provides a useful indication of the procedural complexity to which students are likely to be exposed and the extent to which the majority of students are being challenged beyond the application of procedures. In a study of the use of mathematics textbooks in English, French, and German classrooms, Pepin and Haggarty (2001) analysed how textbooks vary, how they were used by teachers in the classroom and how this influenced the culture of the mathematics classroom. They note that in some textbooks, exercises predominated, with few connections made between the concepts practised. In others, student exploration, questioning, and autonomy were encouraged, and the posing of problems motivated the acquisition of new knowledge. Pepin and Haggarty claim that in the English textbooks "questions were mostly straightforward applications of the worked examples provided. They were the routine-type where a 'taught' method was applied in relatively impoverished and non-real contexts and they only rarely required deeper levels of thinking from pupils" (p. 172). By contrast, they found that the French textbooks contained "graduated exercises with many demanding questions requiring insights and understanding from pupils" (p. 173). In Germany, textbooks were differentiated for the perceived achievement level of students, with a relatively high level of complexity and coherence, particularly with respect to mathematical logic and structure.

Brändström (2005) analysed three different Swedish seventh-grade mathematics textbooks, focusing on how the textbooks provided opportunities for all students to learn. Each book catered for different ability levels by means of two or three alternative strands within each chapter. Brändström's analysis of the textbook tasks included a comparison of the number of operations, the cognitive processes involved (based on Bloom's taxonomy), and the level of cognitive demand on a four-point scale. Brändström found that the lower strands focused predominantly on the lower two levels of cognitive demand (memorisation and applying a procedure). Even in the higher strands, more than 85% of tasks were at the lower two levels. Tasks at the top level were identified only in the strands for more able students in two of the three textbooks, approximately 5% and 10% respectively. It appears, then, that even when textbooks are written specifically for students of different ability levels, only a small proportion of textbook questions challenge students beyond the application of procedures. In view of this literature and the fact that Australian mathematics textbooks are generally written for mixed ability classrooms, the levels of procedural complexity in questions, and the different types of mathematical processes included are important issues.

This paper focuses on the analysis of selected problem sets in a sample of Australian mathematics textbooks, addressing in particular the following research questions:

1. to what extent are the Video Study criteria for procedural complexity, types of mathematical processes, and the exercise/application distinction useful in analysing problem sets and associated tasks in Australian mathematics textbooks?
2. can differences between textbooks be identified using the Video Study criteria?
3. does the analysis of textbook problems align with the findings of the Video Study?

Methodology

In order to gain insight into the methods and findings of the Video Study, we needed to select problems that were typical of Year 8 work, and then analyse them using the Video Study criteria. In this study, we used three of the Video Study variables: procedural complexity, mathematical processes, and the exercise/application classification. Although the Video Study also classified aspects of lesson delivery, the selected variables were applicable to problem statements, and so could be used on textbook problems.

Selecting the Textbooks and Problem Sets

For this preliminary study, we investigated two topics from the 2006 best-selling Year 8 textbooks (textbooks A, B, C, and D) in four Australian states. Each was a clear market-leader. It should be kept in mind that for textbooks A and B, Year 8 is the first year of secondary school, whereas for textbooks C and D, Year 8 is the second year of secondary school. The best-selling textbooks were selected simply because this gave us the best "one book" picture of the problems that might be presented to Australian students. The same topics were also analysed in an additional sample of three different textbooks from one state for which Year 8 is the second year of secondary school (textbooks E, F and G). Because the results were limited to just two topics, and it is unclear whether these are representative, the textbooks are not named in this paper.

All problem sets from two mathematical topics were chosen: *addition and subtraction of fractions* and *solving linear equations*. For solving linear equations, we selected material related to "doing the same to both sides" (not guess and check or graphical solving). These topics were common to all states at this level and were also representative of two of the three most prevalent topic areas in the Video Study – number, geometry and algebra. The problem sets were drawn from the part of the textbook dedicated to that topic. We did not search the rest of the books to find problems that used knowledge from these topics.

Definitions from the Video Study

In each of the selected problem sets, the problems were classified using the Video Study descriptors for procedural complexity, the mathematical processes required in the solution, and as either exercises or applications. Here we describe these classifications.

In the Video Study lesson analysis, problems were defined in the following way: "Problems contain an explicit or implicit Problem Statement that includes an unknown aspect, something that must be determined by applying a mathematical operation, and they contain a Target Result". The Target Result is the answer to the Problem Statement and "may be a number, an algebraic expression, a geometric object, a strategy for solving problems, and even the creation of a new problem" (TIMSS 1999 Video Study Math Coding Manual, pp. 20, 21). A mathematical operation or decision that occurs between the problem statement and the target result is referred to as a *step*. Problems involve one or more steps to reach the target result. Examples of problems provided in the Coding Manual are:

1. Which of the following numbers is bigger?
2. Solve the following equations: (a) $3x + 1 = 8$ (b) $x - 7 = 42$ (2 problems)
3. Find the area of a parallelogram with a base of 8 cm and a height of 4 cm.
4. Make a table of values and graph the equation $3x = 2y - 1$ (problem with sub-problem)

Problems were categorised as either *exercises*, that is, practising a procedure on a set of similar problems, or *applications*, where students applied procedures they had learned in one context to solve problems about a different context. An example of an application problem based on the practised procedure of solving equations is: "The sum of three consecutive integers is 240. Find the integers." (Hiebert et al., 2003, p. 90). Under this definition, applications do not necessarily have real-world references. Problems were classified as being of low, moderate, or high procedural complexity according to the number of steps and sub-problems. The criteria and an example for each level of procedural complexity are shown in Table 3.

Table 3
TIMSS Classification for Problem Complexity [from Hiebert et al., 2003, p. 71]

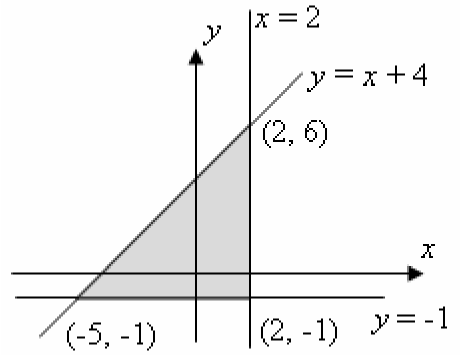
Complexity	Description
Low	Solving the problem, using conventional procedures, requires four or fewer decisions by the students (decisions to be considered small steps). The problem contains no sub-problems or tasks embedded in larger problems that themselves could be coded as problems. Example: Solve the equation: $2x + 7 = 2$
Moderate	Solving the problem, using conventional procedures, requires more than four decisions by the students and can contain one sub-problem. Example: Solve the set of equations for *x* and *y*: $2y = 3x$; $2x + y = 5$
High	Solving the problem, using conventional procedures, requires more than four decisions by the students and contains two or more sub-problems. Example: Graph the following linear inequalities and find the area of intersection: $y \le x + 4$; $x \le 2$; $y \ge -1$

As a check that we were applying the criteria in the intended way, we classified examples including those in Table 3 according to the Video Study criteria. As shown in Tables 4a and 4b, our classifications of complexity coincided with that of the Video Study, although we do not know if steps we identified coincided precisely with those identified by the Video Study, as their steps were not made explicit in the examples.

Table 4a
Examples of Applying the Video Study Criteria for Procedural Complexity

Example 1: Solve the set of equations for *x* and *y*: $2y = 3x$; $2x + y = 5$	
$2y = 3x$ (1) $2x + y = 5$ (2) $4x + 2y = 10$ (3)	Step 1: Decide on an appropriate strategy Step 2: Multiply equation (1) by 2 to get equation (3)
$4x + 3x = 10 \quad \therefore 7x = 10$	Step 3: Substitute equation (1) into equation (3)
$\frac{7x}{7} = \frac{10}{7}, \ x = \frac{10}{7}$	Step 4: Divide both sides by 7
$2y = 3x$ $2y = \frac{3\times 10}{7} = \frac{30}{7}$	Step 5: Substitute $x = \frac{10}{7}$ in equation (1)
$y = \frac{15}{7}$	Step 6: Divide both sides by 2
More than four steps, but no sub-problem, so moderate procedural complexity.	

Table 4b
Examples of Applying the Video Study Criteria for Procedural Complexity

Example 2: Graph the following linear inequalities and find the area of intersection: $y \le x + 4$; $x \le 2$; $y \ge -1$	
x-intercept (-4, 0), *y*-intercept (0, 4) 	Step 1: Find intercepts for $y \le x + 4$ Steps 2- 4: Sketch graphs Sub-problem: Steps 5, 6: Find coordinates of intersections Step 7: Decide on required region Sub-problem: Steps 8, 9: Find base and height of right-angled triangle Step 10: Calculate area of triangle
More than four steps, and two sub-problems, so high procedural complexity.	

Problem statements were also categorised according to the implied mathematical processes: *using procedures*, *stating concepts,* or *making connections*. The criteria and an example for each category are shown in Table 5.

Table 5
Defining the Types of Mathematical Processes Implied by Problem Statements [from Hiebert et al., 2003, p. 98]

Mathematical process	Description
Using procedures	Problem statements that suggested the problem was typically solved by applying a procedure or set of procedures. These include arithmetic with whole numbers, fractions, decimals, manipulating algebraic symbols to simplify expressions and solve equations, finding areas and perimeters of simple plane figures, and so on. Example: Solve for x in the equation $2x + 5 = 6 - x$.
Stating concepts	Problem statements that called for a mathematical convention or an example of a mathematical concept. Examples: Plot the point (3, 2) on a coordinate plane. Draw an isosceles triangle.
Making connections	Problem statements that implied the problem would focus on constructing relationships among mathematical ideas, facts or procedures. Often, the problem statement suggested that students would engage in special forms of mathematical reasoning such as conjecturing, generalizing, and verifying. Examples: Graph the equations $y = 2x + 3$, $2y = x - 2$ and $y = -4x$, and examine the role played by the numbers in determining the position and slope of the associated lines.

Results

Variations occurred in the way the fractions problems were organised, with some textbooks including addition and subtraction together in a single problem set, and others presenting addition and subtraction separately. In some books, simple fractions were placed in a separate problem set from mixed numbers. In one book, students were directed to use calculators in the problems involving mixed number addition and subtraction. In states where Year 8 was the first year of secondary schooling, the Year 8 textbooks (A and B) included an extensive treatment of fractions, compared with the states where Year 8 was the second year of secondary schooling. Although textbook E provided substantial revision, textbooks C, D, and F included only a small number of problems and G had no fractions section. Textbook D focused only on very simple problems with no mixed numbers.

Table 6 shows the number of problems, procedural complexity, and type of solving process for "Addition and subtraction of fractions" problems in the sample of seven textbooks. The majority of problems in all books were of low complexity. The data in Table 6 show a tendency for the textbooks that regarded this as a revision topic to have relatively more problems of moderate complexity, although textbook D is an exception. Almost all of the problems required only *using procedures*. Although the relatively high percentage of *making connections* problems in textbook C represents only four problems from a small revision set, it does indicate a different approach to this revision than in the other books.

A similar pattern of procedural complexity was found in the problems relating to solving linear equations (see Table 7). One might expect that in states where Year 8 was the second year of secondary schooling a smaller percentage of low complexity problems would appear in the Year 8 textbooks. However, this was not the case. Textbook C, for example, contained the highest proportion of low complexity problems despite the inclusion of equation solving in the corresponding Year 7 book. All textbooks included at least some problems that required students to make connections (ranging from 2% for textbook A to 27% for textbook B) but the focus was still predominantly on using procedures. Wide variation in the number of problems was also evident, ranging from 87 problems in textbook A to 337 problems in textbook D. The last line of Tables 6 and 7 gives the Australian averages from the Video Study for comparison. The lessons of the Video Study had more problems of high complexity and more problems requiring stating concepts and making connections than these two sections of the textbooks.

Table 6
Procedural Complexity and Type of Solving Process for "Addition and Subtraction of Fractions" Problems for Sample of Australian Year 8 Mathematics Textbooks

Textbook	Number of problems	Procedural complexity (percentage of problems)			Solving process (percentage of problems)		
		Low	Moderate	High	Using procedures	Stating concepts	Making connections
A	114	76	24	0	93	2	5
B	116	76	24	0	95	0	5
C	16	56	44	0	75	0	25
D	12	83	17	0	100	0	0
E	74	69	31	0	95	0	5
F	18	61	39	0	100	0	0
G	no section	-	-	-	-	-	-
Video 99 Australia		77	16	8	61	24	15

Note: The percentages do not all sum to 100 because of rounding.

Table 7
Procedural Complexity and Type of Solving Process for "Solving Linear Equations" Problems for Sample of Australian Year 8 Mathematics Textbooks

Textbook	Number of problems	Procedural complexity (percentage of problems)			Solving process (percentage of problems)		
		Low	Moderate	High	Using procedures	Stating concepts	Making connections
A	87	79	21	0	84	14	2
B	132	85	15	0	64	9	27
C	213	88	12	0	72	5	23
D	337	85	15	0	91	1	8
E	298	73	26	1	89	1	10
F	172	62	38	0	77	7	16
G	250	84	16	0	91	4	6
Video 99 Australia		77	16	8	61	24	15

Classification of the problems as either exercises or applications indicated that the emphasis for both topics in all textbooks was on the practising of procedures (exercises) rather than on the application of those procedures (see Figure 1). This was particularly evident in the case of addition and subtraction of fractions, where only three books included application problems. Curiously, books C, D, and F, which were revising the topic from the previous year's work (recall that G had no section on this topic), had no application problems. It will be interesting to check with other topics whether revision focuses more strongly on procedures than is the case when the topics are first introduced. (Note that there is a methodological difficulty here that urges caution: only problems in the designated chapters have been analysed, but there may be many applications in later chapters).

For solving linear equations, the average proportion of application problems was higher (see Figure 1) with more variation. Textbook A had only exercises, but most books included a number of application word problems, sometimes in separate problem sets or as investigations. In textbook B (from a state where Year 8 is the first secondary school year) over 25% of the problems were applications involving the solving of word problems. However, when the total number of equation solving problems was considered, it could be seen that there was a high level of repetitive exercises. In textbook D, for example, there were only 43 application problems from a total of 337 problems.

Two further observations are of interest. First, the relative proportions of applications and exercises in the books vary between the two topics. It does not appear that some books have more applications in all chapters. Second, the proportions of applications for all of these textbooks for both topics are substantially below the Australian average of 45% of problems being applications in the Video Study lessons.

Discussion

As in the Video Study, textbook problems were overwhelmingly low complexity problems and they focussed on using procedures. There was a broad similarity in the proportions of problems in each category in this and the Video Study, although it will be useful to test this on a further sample of textbooks and topics. In fact, the results of the Video Study showed more variation than the textbook problems, having more problems of high complexity, more applications and fewer problems that only required using procedures. This may indicate that much of this variation in lessons came from resources other than textbooks, and the Video Study data can be examined in future to test this.

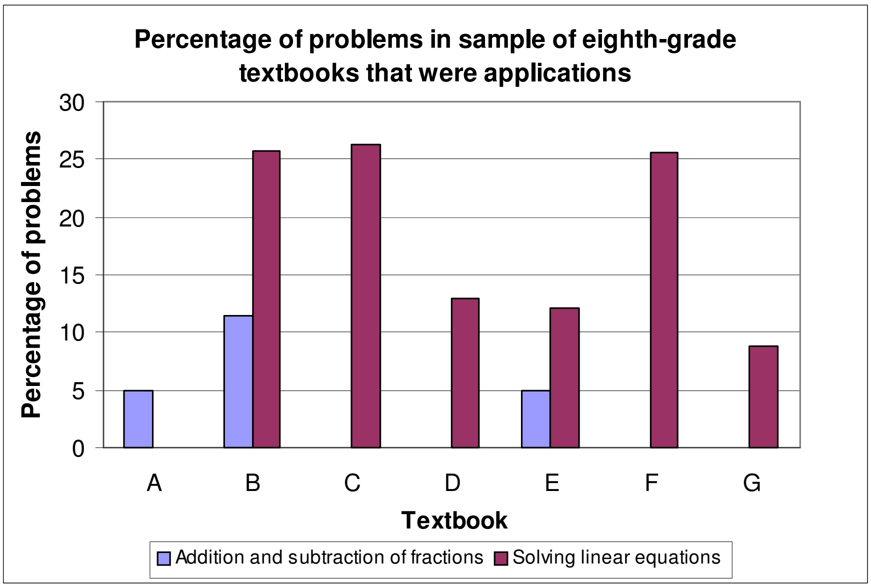


Figure 1. Percentage of problems in the sample of Year 8 textbooks that were applications.

Choosing topics that were comparable across the different states was complicated by the slightly different curriculum emphases, and by whether Year 8 was the first or the second year of secondary schooling. Although addition and subtraction of fractions was a common curriculum element, in states where Year 8 was the second year of secondary school, most textbooks included only a brief revision set of problems. However, contrary to expectations, these revision problems were generally low complexity exercises, with few application problems or problems that required students to make connections or consider underlying mathematical concepts. Consequently, students with conceptual difficulties after first exposure to a topic are less likely to have them addressed in later years.

A major aim of this study was to explore the use of the definitions and constructs of the Video Study, and their suitability for capturing the essence of the mathematical work on which students spend their time. In general, the classification procedures seemed reasonably robust. For example, in determining problem complexity, it was sometimes difficult to decide whether to count a particular operation as one or two steps. At Year 8 level, students are likely to be still gaining confidence with addition and subtraction involving negative integers. Hence we classified solving the equation $-2x-5=-11$ as requiring three steps: deciding to add 5 to both sides, calculating $-11+5$, and dividing both sides by -2. However, the equation $2x+5=11$ was classified as having only two steps: subtracting 5 from both sides to give 6 on the right side, and dividing both sides by 2. With either 2 or 3 steps, though, both these equations are classified as low complexity.

Different types of problems play different pedagogical roles. It is important that textbooks provide students with sufficient exercises so that procedures may be practised and become a secure part of a student's mathematical toolbox. Likewise there should be sufficient problems for students to learn to apply those practised skills, for making connections between different aspects of mathematics, for recognising underlying mathematical concepts, and for reasoning. Having two classifications, one for complexity and one for mathematical processes, highlights the fact that higher procedural complexity does not indicate higher quality of problems in terms of challenging students to make connections or to reason. In the case of the equation solving problems, for example, many problems qualified as moderate complexity because the solving required more steps, for example, $7(x-3)-2(5-x)+25=4(x+3)-8$. However, apart from deciding upon the order of steps, the student simply repeats the same types of operations: expanding brackets, dealing with positive and negative signs, collecting like terms, etc. It is important that students should be able to solve equations involving multiple steps. Mathematicians have to be able to sustain a chain of reasoning without error. However, the textbooks tended to include these moderately complex equations at the expense of including high complexity problems, where students must plan a path through sub-problems in order to reach the target result. In several books, investigations were included that would have been classified as one high-complexity problem, except that the investigation was broken down into a number of clearly stated sub-problems, each of which became a separate problem of generally low complexity.

It was also evident during the classification process, that the classifications do not show which are “good” problems, and that there are problems that provoke and do not provoke mathematical thought in all categories. A problem such as “plot the point (3, 2)”, for example, is classified as “stating concepts”, but it may stimulate less learning than a simple “using procedures” problem. It is not that “using procedures” problems and problems of low complexity are “bad” of themselves, but that their dominance curtails the experiences that students have of mathematical thinking. It is also the case that using the percentage of problems in each category as the basic measure is problematic (providing a few more exercises will put up the percentage of low complexity problems), especially as problems of higher complexity and those requiring connections may each take more students’ time.

References

Brändström, A. (2005). *Differentiated tasks in mathematics textbooks: An analysis of the levels of difficulty*. Licentiate Thesis, Luleå University of Technology, Sweden.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., Chui, A. M-Y., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng., E., Etterbeck, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 Video Study*. Washington, DC: National Centre for Education Statistics, U.S. Department of Education.

Pepin, B., & Haggarty, L. (2001). Mathematics textbooks and their use in English, French and German classrooms: A way to understand teaching and learning cultures. *Zentralblatt für Didaktik der Mathematik, 33*(5), 158 – 175.

Stacey, K. (2003). The need to increase attention to mathematical reasoning. In H. Hollingsworth, J. Lokan & B. McCrae, *Teaching Mathematics in Australia: Results from the TIMSS 1999 Video Study*. (pp. 119-122) Melbourne: Australian Council for Educational Research.