Whole Number Knowledge and Number Lines Help to Develop Fraction Concepts

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Many researchers have noted that students' whole number knowledge can interfere with their efforts to learn fractions. In this paper we discuss a teaching experiment conducted with students in Years 5 and 6 from an eastern suburban school in Melbourne. The focus of the teaching experiment was to use number lines to highlight students' understanding of whole numbers then fractions. This research showed that successful students had easily accessible whole number knowledge and recognised the relationship between the whole and the parts whereas the weakest students had poor number knowledge and could not see the connections.

Research Background

Over the past 20 years research on rational number learning has focused on the development of basic fraction concepts. This has included partitioning of a whole into fractional parts, naming of fractional parts, and order and equivalence (Behr, Wachsmuth, Post, & Lesh, 1984; Kieren, 1983; Streefland, 1984). Kieren (1976) distinguished seven interpretations of rational number that were necessary to enable the learner to acquire sound rational number knowledge, but subsequently (Kieren, 1980; 1988) condensed these into five: whole-part relations, ratios, quotients, measures, and operators. Kieren suggested that difficulties experienced by children solving rational number tasks arise because rational number ideas are sophisticated and different from natural number ideas and that children have to develop the appropriate images, actions, and language to precede the formal work with fractions, decimals, and rational algebraic forms.

Several researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions (Behr et al., 1984; Bezuk, 1988; Hunting, 1986; Streefland, 1984). Behr and Post (1988) indicated that children need to be competent in the four operations of whole numbers, along with an understanding of measurement, to enable them to understand rational numbers. They suggested that rational numbers are the first set of numbers experienced by children that are not dependent on a counting algorithm. The required shift of thinking causes difficulty for many students.

Mack (1990) found that where students possessed knowledge of rote procedures they focused on symbolic manipulations. Mack's study suggested that if a strand of rational number is developed based on partitioning, using the students' informal knowledge, then other strands of rational number could be developed more easily.

Steffe and Olive (1990) showed that concepts and operations represented by children's natural language are used in their construction of fraction knowledge. Two distinct fraction schemes emerged from their research. In the iterative scheme, children established a unit fraction as part of a continuous but segmented unit. From this, children developed their own fraction knowledge by iterating unit fractions. The foundation of a measurement scheme occurred when the children's number sequence was modified to form a connected number sequence.

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Saenz-Ludlow (1994) maintained that students need to conceptualise fractions as quantities before being introduced to standard fractional symbolic computational algorithms. Streefland (1984) discussed the importance of students constructing their own understanding of fractions by constructing the procedures of the operations, rules, and language of fractions. This research focuses on students' use of number lines firstly to probe students' understanding of fractions as numbers capable of being represented on a number line, and then to look at how number lines involving whole numbers and fractions can be used to develop fractional language and to articulate fractional concepts.

Previous Studies

In previous research (Pearn & Stephens, 2004; Pearn, Stephens, & Lewis, 2002; Stephens & Pearn, 2003) analysis of results from the Fraction Screening Test A (Pearn & Stephens, 2002) has highlighted students' difficulties with fraction concepts. The Fraction Screening Test is a paper and pencil test designed mainly for students in Years 5 and 6 and for weaker students in Years 7 and 8. The tasks include contexts such as discrete items, lengths, fraction walls, and number lines. Analysis of the results from the Fraction Screening Test highlighted the difficulties that many students experienced with number lines. The three number line tasks from the Screening Test are shown in Figure 1.

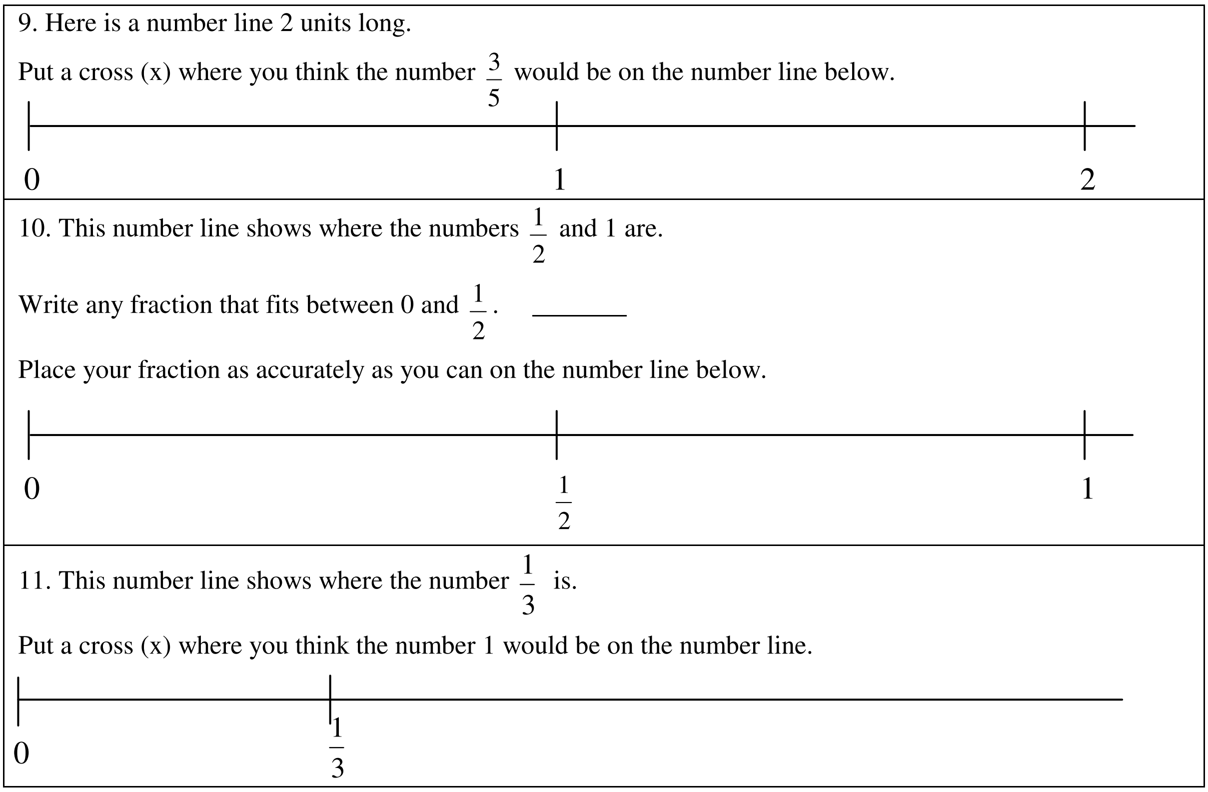


Figure 1. The three number line tasks (Fraction Screening Test A).

Many students in Question 9 confused three-fifths of the number line with the number three-fifths. In Question 10 many students who chose one-quarter represented it correctly. Other fractions seemed to be placed using guess work rather than any systematic division of the number line. A similar tendency to use guess work was evident in Question 11. Table 1 compares the results of 288 students in four year levels from four different Victorian schools on the above three number line tasks. These results highlight the difficulties that students have with the notion of fractions as numbers and with placing the fractions on number lines accurately.

Table 1
Success with Tasks from the Fraction Screening Test A (n = 288)

Task from Fraction Screening Test A	Year 5 (n = 84)	Year 6 (n = 66)	Year 7 (n = 89)	Year 8 (n = 49)
Marks 3/5 of the number line	23%	32%	32%	20%
Chooses then marks number between 0 and 1/2	44%	31%	52%	19%
Marks 1 given 1/3	46%	41%	56%	25%

Subsequent Interviews

In a previous study (Pearn & Stephens, 2004), several students who had completed the Screening Test were asked to compare two fractions and then place them on number lines marked zero to one. We observed that some students just “placed” the fractions on the number lines without using any referents to other known fractions, for example, one-half. For example, one student randomly placed the fraction three-quarters close to the number one on the number line then placed three-fifths the same distance from three-quarters as she had placed three-quarters from one (Figure 2). This was because, “three-quarters is only one away from a whole and three-fifths is two away from a whole”. Pearn and Stephens (2004) refer to this as gap thinking, illustrating how whole number thinking can interfere with fraction knowledge.

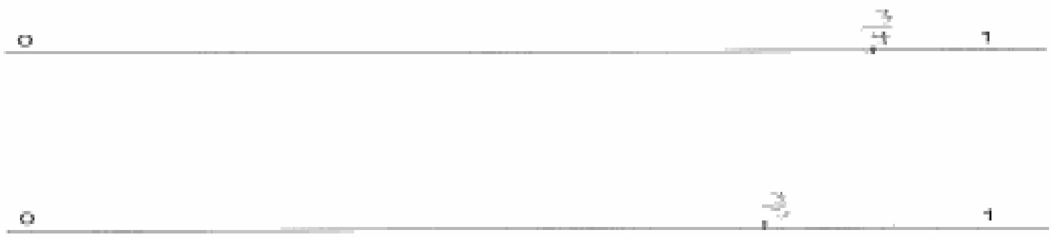


Figure 2. Three-quarters and three-fifths.

Another student when comparing three-quarters and three-fifths correctly converted both fractions to twentieths concluding that three-quarters was bigger (Pearn & Stephens, 2004). When invited to use number lines to compare these two fractions he divided the first number line (below) by eye into quarters and marked one-half and three-quarters. He then placed one-half on the number line below corresponding to its position on the first number line. He said that “three-fifths is smaller than three-quarters” and marked three-fifths to the right of one-half and to the left of three-quarters on the first number line with no attempt to divide the line into fifths (Figure 3).

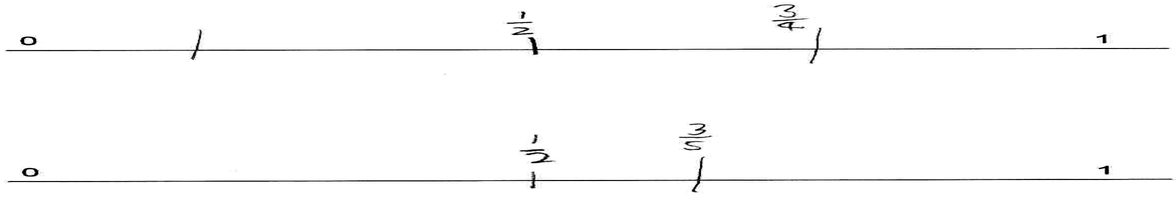


Figure 3. One-half, three-quarters and three-fifths.

When the interviewer asked where the fraction one-fifth would be the student responded with “One-fifth is more than one-half, I think.” He then used a new number line and placed one-fifth to the right of one-half. The interviewer then asked where he thought one-third and one-quarter would be on the number line. The student then placed these two fractions in between one-half and one-fifth as shown in figure 4. Despite apparent correct thinking in the previous example, this student unexpectedly lapsed into *larger-is-bigger thinking* – another example of incorrect whole number thinking.

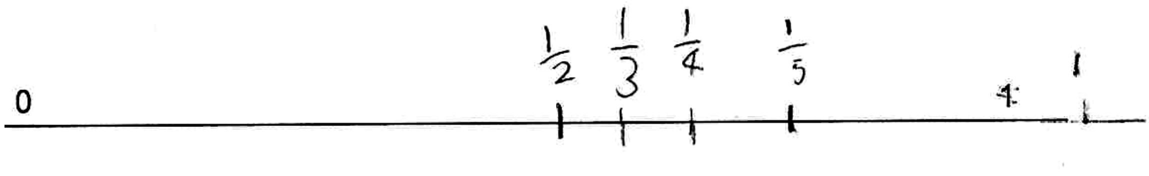


Figure 4. Larger denominator is bigger.

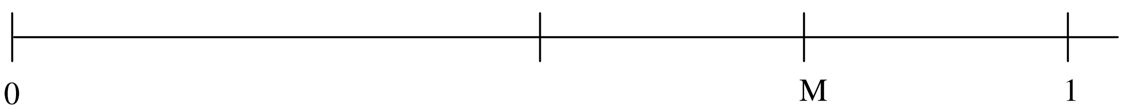
These instances demonstrate the importance of asking students in a probing interview to represent their fractional thinking using a number line. On the other hand, asking other students to represent fractions on a number line assisted them to identify and correct their misconceptions. However the study did not set out to explore remedial strategies with the students interviewed.

The current study also uses a screening test and interview using number lines to probe students’ understanding of fractions as numbers. The interview commenced by looking at how number lines involving whole numbers can be used to develop fractional language and to articulate fractional concepts.

Initial Testing

All students from Years 5 and 6 from School A in the eastern suburbs of Melbourne were given Fraction Screening Test A (Pearn & Stephens, 2002). The tasks used contexts such as discrete items, lengths, fraction walls, and number lines. One fraction task based on area was replaced in this study with an extra number line task. Figure 5 shows the additional number line task added specifically for this group of students.

12. On this number line 0 and 1 are shown.



What fraction number do you think M represents? _______

Figure 5. Additional number line task (Fraction Screening Test A).

Results

The students’ results on the Fraction Screening Test A reflected the types of responses achieved previously from other groups of students. Results shown in Figure 6 show that these students were more successful with tasks presented in conventional contexts such as shading three-fifths of an unmarked rectangle and with the fraction one-third, for example, finding the whole given a third using discrete objects. They were less successful with tasks that involved fractions as numbers, for example “Put a cross (x) where you think the number $\frac{3}{5}$ would be on the number line". Many students interpreted this question as requiring them to find three-fifths of the entire line ignoring the numbers 0, 1, and 2 marked on the number line.

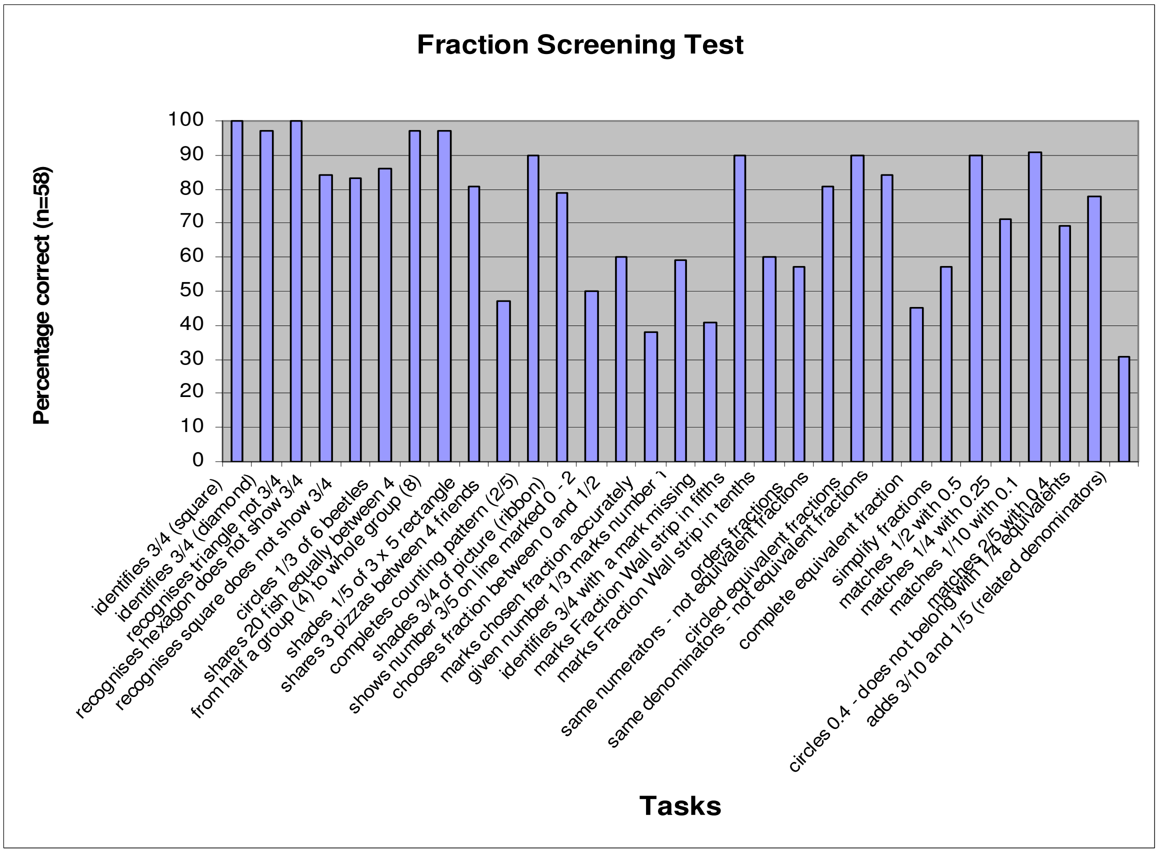


Figure 6. Success with tasks from Fraction Screening Test A.

Teachers from School A had undertaken considerable professional development presented by the authors. In Table 2 we compared the combined results of Years 5 and 6 in School A with results on the same three questions from other schools (see Table 1) where teachers had not had the same level of professional development. Students at School A were more successful with the first and third tasks. In the second task, while 60% of School A's students were able to state a fraction between 0 and ½, only 38% could place the fraction they chose accurately on the number line.

Table 2
Comparative Success of Students from School A on Fraction Screening Test

Number line tasks (Fraction Screening Test A)	*School A (n = 58)*	*Other Year 5 (n = 84)*	*Other Year 6 (n = 66)*
Marks 3/5 of the number line	*50%*	23%	32%
Chooses then marks number between 0 and 1/2	38%	44%	31%
Marks number 1 given 1/3	*59%*	46%	41%

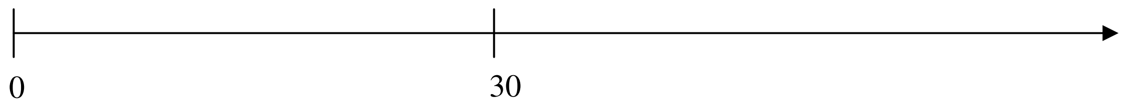
Analysis of the additional number line question (Figure 5) revealed that only 41% of the students from School A were able to identify the number denoted by M (3/4) on the number line. A few students thought the letter M should represent a letter so responses included words like "million", "middle", and "mixed number".

Fraction Number Line Interview

The authors developed an interview protocol called *Working with number lines to probe fraction concepts* (Pearn & Stephens, 2006). The interview required students to complete number line tasks while describing what they were thinking or how they worked it out. Students were initially required to place whole numbers on number lines, then fractions on number lines and finally, to review their responses to the four number line questions from the Fraction Screening Test. Figure 7 is an example of one question that requires students to place a number between two given whole numbers and then place another number relative to one of the given whole numbers. Following research by Behr and Post (1988) and Mack (1990), questions like this were designed to see how well students could connect their whole number knowledge in a fraction context.

4. This number line shows 0 to 30.

a) Where would you put the number 10? How could you be sure?



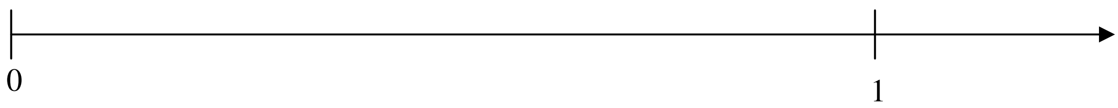
b) If you want to mark the number 40 on this line how could you do that? How could you be sure?

Figure 7. Marking whole numbers on a number line.

After working with whole numbers students were asked to place proper fractions and mixed numbers on number lines. Figure 8 gives an example of a question involving fractions. For this task the interviewers were looking for evidence that students could place fractions accurately by using points of reference rather than just "placing" the fraction randomly on the line. The second part of this task requires students to use previous information to assist them to decide the most appropriate point for the number.

8. This number line is marked 0 and 1.

a) Where would you put the number ¾? How could you be sure?



b) If you want to mark the number $1\frac{1}{8}$ on this line how would you do that? How could you be sure?

Figure 8. Marking fractions on a number line.

The Interviews

Students were individually interviewed by the authors. In Task 1, students were shown a number line marked 0 and 100. They were then asked to show where the number 50 would be placed. Students justified their answers by saying things like:

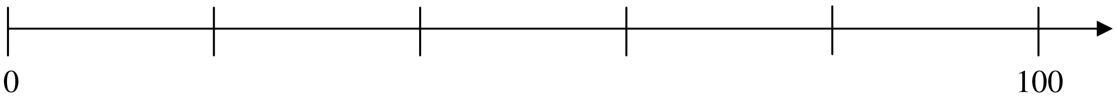
- 50. It's in between. Half of a 100 is 50.

- Another student placed 50 correctly and said: "It's in the middle (of the line)."

Many students found Task 2 (Figure 9) more difficult where, unlike the previous task, the midpoint of the line was unmarked. Some students' responses to this task highlighted the lack of understanding of the relationship between the number of marks used to divide the line and the numbers parts so formed. Despite giving a correct answer, Student S could not connect her numbers to the parts. Even when students, like Students R and T, were helped to identify the number of parts their lack of number knowledge prevented them from giving a confident correct response.

2. This number line is marked 0 to 100 and has been divided up.

Can you work out what numbers should go on the marks?



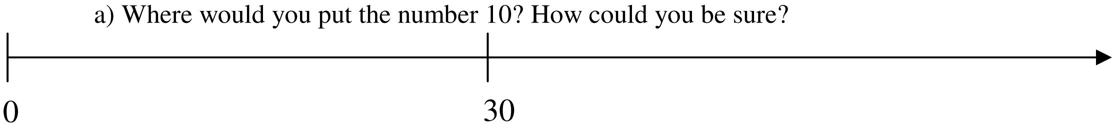
Student R (Year 5)	Student S (Year 5)	Student T (Year 5)
Pointing to the last mark (where 80 should be) she said: "Maybe this should be 75". Interviewer: How many parts? A: four … six I: Count the parts. A: Five I: Five people to share 100 lollies. How many each? A: Fifteen … 15, 30, 45, 60 … No. Maybe 30 … maybe 25 … maybe 20 I: Please check for 20. A: correctly marked the line 20, 40, 60 … to 100	Wrote 20, 40, 60, 80 S: I just know. I: How many parts are there? S: Four I: Does it help to know the parts? S: Not really.	Placed the numbers 15, 20, 60, 75 on the marks provided. I: Is 15 going to work? A: 15, 30, 45, 60 … No. He then placed 20, 40, 60, 80. I: How many spaces? A: Five I: Share 100 between five people. I: 2 … 20

Figure 9. Examples of students' responses for Task 2.

Those students who knew that 30 consisted of three 10s, or that 10 was one third of 30, dividing the number line into three equal parts was easy. For students like Student T the process of halving and then partitioning again proved problematic (see Figure 10).

4. This number line shows 0 to 30.

a) Where would you put the number 10? How could you be sure?



b) If you want to mark the number 40 on this line how could you do that? How could you be sure?

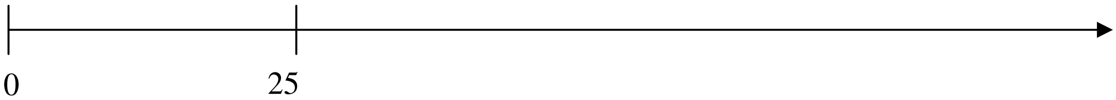
Student S: About there. (Placing 10 correctly). It's about a third. I: "Could you check." She marked in 10, 20, and 30 correctly. Placed 40 correctly. "Because it's the same distance (10) up from 30".	Student T first placed 15 half way. Then said: "Twenty would be about there." He then estimated where 10 would be (no partitioning) and decided that 40 would be the same distance from 30.

Figure 10. Examples of students' responses for Task 4.

For Task 5, (Figure 11), several students, including Students U and V, assumed the arrow at the end of the drawn line was the mark for 100. These students used this assumption rather than the information given on the number line.

5. This number line shows 0 to 25.

a) Where would you put the number 75? How could you be sure?



b) If you want to mark the number 5 on this line how could you do that?

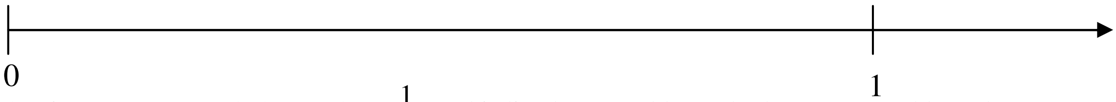
Student S Put in two marks to represent 50 and 75 but very inaccurate increments of 25. I: Could you use your pencil to measure? A used an accurate measure to place 50 and 75 but didn't know how to place 5.	Student U marked 50 then 75. "I think here is about 100 (end of drawn line). "Three-quarters is 75. Because 25, 50, 75". Placed 5 about half way between 0 and 25, then rethought. I: Half of 25? S: 12½ I: Half of 12 is …? S: "Six". Placed 5 a bit to the left of where 6 would be.	Student V (Year 6) marked in two more intervals to correctly place 75. He appeared puzzled because he assumed the end (arrow) was 100. He initially subdivided 0 to 25 too small. Self corrected to get fifths quite accurately.

Figure 11. Examples of students' responses for Task 5.

In Figure 12 the interviewer assisted students by asking them to focus on the interim fractional points (¼, ½, and ¾). Some students thought the arrow was the mark for the number two but once they had focussed on the interim fractional points were able to correctly place 1⅛ by subdividing correctly the line between 1 and 1¼.

8. This number line is marked 0 and 1.

a) Where would you put the number ¾? How could you be sure?



b) If you want to mark the number $1\frac{1}{8}$ on this line how would you do that? How could you be sure?

Student S marked ½ then ¾ correctly by eye. Not sure about 1⅛. I: What's the distance between ¾ and 1? S: ¼ I: Where is 1¼? She identified 1¼ and then said "Half of that (distance between 1 and 1¼) is 1⅛".	Student T said: ¾ is about here (placed it but didn't use ½ or ¼ as reference points). I: Where is ½ and ¼? He subdivided and then was able to place 1¼ correctly and halved the distance from 1 to 1¼ to get 1⅛

Figure 12. Examples of student responses for Task 8.

Analysis of Interview Results

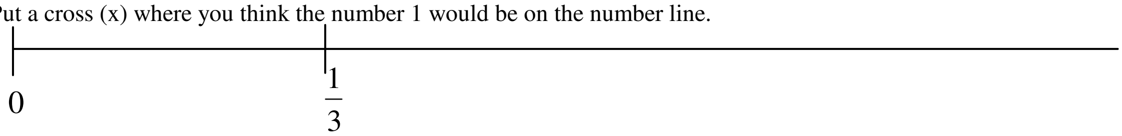
Successful students used number knowledge, accurate skip-counting, and multiplication facts to partition the number line. They confidently related halves, quarters, and three-quarters to the numbers being used. For example they could relate eighths to quarters. Some students needed help to identify the number of spaces (parts) instead of focussing only on the vertical division marks. The number line questions allowed those students who had confident whole number knowledge to apply fractional concepts to their subdivisions of the number line. Other students who were unable to draw on whole number knowledge frequently used guesses to place numbers on the number line using "Where I think it should be" rather than accurate "by-eye" partitioning. These students were rarely able to apply the language of fractions to subdivisions of the number line, and often needed assistance to see connections between halves, quarters, and eighths.

Students Reviewing their Written Responses to the Screening Test

On the initial Screening Test, Student S correctly marked the number one but showed no evidence of the strategy she used. Student T's response showed no understanding of equal intervals. However after being interviewed Students S and T applied correct subdivision strategies to this task that they had used for their whole number questions (Figure 13).

This number line shows where the number $\frac{1}{3}$ is.

Put a cross (x) where you think the number 1 would be on the number line.



On the Screening Test Student S had placed the number 1 correctly with no interim marks. In interview she doubled 1/3 to give 2/3 and added a further 1/3 to get 1 then explained that she could also count in sixths by halving the line 0 to 1/3.	On the Screening Test Student T had placed the number one incorrectly (too close with two incorrect interim marks that were not equivalent to 1/3). I: "Is your old one correct?" He then used the distance from 0 to 1/3 to create 2/3 and added 1/3 accurately to get 3/3.

Figure 13. Comparison of Task 3 responses before and after the interview.

When asked to review their earlier written responses, many students showed evidence of being able to recognise errors and to self correct, as shown in Figure 14, for the fraction task using the letter M. Both Students S and T were now able to see that the letter M represented the fraction ¾.

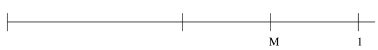
Task	Student S	Student T
On this number line 0 and 1 are shown. What fraction number do you think M represents? 	S wrote ¼ as the value of M on the Screening Test. In interview she said: "M is ¼ … ? Oh no, it's ¾."	A wrote that M was 2/3 but in interview he said: "It should be ¾." I: Why did you choose 2/3? A: Because there were three parts. He then added by pointing: "That part (0 → ½) is bigger than this (½ → 2/3)"

Figure 14. Comparison of Task 4 responses before and after the interview.

Conclusions

Successful students demonstrated easily accessible and correct whole number knowledge and knew relationships between whole and parts. They attended to equal parts not the vertical lines used to create the parts. They could apply fractional terms to the equal parts. Less successful students tended to look at lines and needed help to focus on equal parts. These students often had difficulties with number lines marked without a midpoint. Sometimes these students assumed that arrows at the end of lines represented "the next" whole number. Due to their poor whole number knowledge, the weakest students could not see connections between whole numbers and fractional parts of the number line. Also, they appeared dependent on guess work to place numbers on number lines.

By using whole numbers on number lines first, the interview questions clearly helped many students to connect whole number and fraction knowledge. The interviews also helped students to recognise and correct their own misconceptions in previous assessment tasks.

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