

Identity and Mathematics: Towards a Theory of Agency in Coming to Learn Mathematics

Peter Grootenboer
Charles Sturt University
<pgrootenboer@csu.edu.au>

Robyn Zevenbergen
Griffith University
<r.zevenbergen@griffith.edu.au>

In writing this paper we draw considerably on the work of Jo Boaler and Leone Burton. Boaler's studies of Railside have been particularly poignant in alerting the mathematics education community to a number of key features of successful classrooms, and how such features can turn around the successes for students who traditionally perform poorly in school mathematics. This is supplemented by the more recent work of Leone Burton who worked extensively with research mathematicians in order to understand their communities and ways of working. Collectively these two seminal works provide valuable insights into potential ways to move the field of school mathematics forward. In times where there is international recognition of the plight of school mathematics, there is a need for new teaching practices that overcome the hiatus of contemporary school mathematics.

For a long time now we have known that there have been serious problems with mathematics participation and engagement. The desperate situation facing mathematics has been highlighted recently in Australia by two significant reviews into the mathematical sciences:

1. Statistics at Australian Universities (Statistical Society of Australia, 2005)
2. Mathematics and Statistics: Critical Skills for Australia's Future (Australian Academy of Science, 2006).

Although these reviews were conducted in Australia, a similar story has emerged around the world and it is now approaching a crisis situation. In these reviews, particular attention has rightly been given to school mathematics and the problems of non-engagement with an increasing number of students in higher level courses of mathematical study. That said, we have known for a long time that mathematics has been unpopular and disliked through the many descriptive studies that have been undertaken since the 1970's, and yet the problems appear to grow unabated and little progress has been made to arrest the decline. At this critical point we want to suggest that it is time to move on from studies that repetitively show that mathematics is suffering from a "poor image" and a "lack of friends", and to try and look forward by offering some positive directions to arrest the decline. To advance this agenda we need more than good ideas that seemed to have worked in a particular context; we need to begin developing a theoretical, robust framework that will address these concerns in a coherent and holistic fashion. In this paper we have drawn on the seminal works of Burton and Boaler to consider mathematical learning from both the discipline knowledge and the mathematical activity perspectives. After reviewing Burton's findings from her study with research mathematicians we briefly highlight some relevant points from Boaler's study of Railside. After presenting an example from teacher education we finish by employing the metaphor of a "dance of agency" (Pickering, 1995) to discuss mathematics learning, particularly in the light of the current crisis.

The Practice of Mathematicians

The two recent reviews of mathematical sciences in Australia mentioned earlier both made significant comment and recommendations for school mathematics education. Interestingly, the authors of these reports were mathematical scientists and there appeared to be little input from mathematics educators and mathematics teachers. Although this is problematic, it does perhaps highlight the gap that seems exist between mathematicians and statisticians, and teachers and educators. This is unhealthy and if the current decline in participation and interest in mathematics is to be arrested then these groups need to engage in dialogue and mutual projects. To this end, the work of Burton (1999, 2001, 2002) is helpful because her research explored the practices of research mathematicians and their implications for the learning of mathematics.

In 1997, Burton studied the practices of 70 research mathematicians in Great Britain and one of the key features she identified was the collaborative nature of their practice. The benefits for collaborating included practical (e.g., sharing the work), quality (e.g., greater range of ideas on problems), educational (e.g., learning from one another) and emotional (e.g., feeling less isolated) reasons. Clearly working together with other mathematicians was seen as important, but there appeared to be a distinction between the public perception of mathematics as a lonely enterprise and the reality of mathematicians' practice where collaboration is highly valued.

Perhaps another anomaly from public perception was Burton's finding that mathematicians have emotional, aesthetic, and personal responses to mathematics.

... although knowing when you know is extremely important, you have to live with uncertainty. You gain pleasure and satisfaction from the feelings that are associated with knowing. These feelings are exceptionally important since, often despite being unsure about the best path to take to reach your objective, because of your feelings you remain *convinced* that a path is there. ... This is particularly poignant in the light of the picture painted of mathematics as being emotion-free ... (Burton, 1999, p. 134)

The mathematicians in her study highlighted the power of the “aha!” moment and the joy of mathematical discovery, revealing the clear link between mathematics and those who produce it. Allied to their emotional responses to their mathematical practice were aesthetic reactions. They described mathematics in terms such as “wonder”, “beauty”, and “delight” and these personal responses provided motivation for continued engagement and fuelled a passion for the discipline of mathematics. Davis and Hersch (1998, p. 169) lamented that “blindness to the aesthetic element in mathematics is widespread and can account for the feeling that mathematics is dry as dust, as exciting as a telephone book ...”.

Another feature of research mathematicians practice was the importance of intuition or insight. Although the mathematicians were less than clear in describing what intuition and/or insight were, they were unambiguous in highlighting their importance in their mathematical practice. The suggestion was that intuition can be developed through the application of knowledge and experience in mathematical discovery and reflection upon such investigations.

Burton highlighted other features of the practice of mathematicians including the desire to seek and see rich connections between the various branches of mathematics and between mathematics and other disciplines, but her other main agenda was to highlight the pedagogical implications of her findings. Throughout her reports Burton highlights the distinction that is evident between the work and learning practices of research mathematicians, and the learning experiences of mathematics students at almost all other

levels from preschool to undergraduate degrees. This led her to assert that “we have a responsibility to make the learning of mathematics more akin to ho” in the absence of any student’s need to know” (Burton, 2001, p. 598). Even at a very general level, this would require mathematical pedagogy to be characterised by collaboration and group work with attention paid to the emotional, aesthetic, and intuitive dimensions of the discipline. This encompasses the “doing” of mathematics that has been under-emphasised in education as it has focussed on the “knowing” of mathematics. Indeed, perhaps an issue with the educational recommendations in the Australian review of mathematical sciences was the emphasis on mathematical content knowledge that can be taught largely through a transmission model. On this point Boaler (2003) commented:

There is a widespread public perception that good teachers simply need to know a lot. But teaching is not a knowledge base, it is an action, and teacher knowledge is only useful to the extent that it interacts productively with all the different variables in teaching. Knowledge of subject, curriculum, or even teaching methods, need to combine with teachers’ own thoughts and ideas as they too engage in something of a conceptual dance. (p. 12)

In her seminal work in England, Boaler (1997) explored the mathematical practices of teachers and students in two different sorts of mathematics classrooms. In one group of classes, the mathematical pedagogy was “traditional” and the students learned standard algorithms through worked examples and textbook exercises. The other classrooms were characterised by open-ended projects, group work and discussion. Not surprisingly, she found that the students by and large learned a form of mathematics that was consistent with the mathematical epistemology and pedagogy of their classroom experiences. However, in general the students in the “non-traditional” classes performed better in a range of assessment tasks and overall they developed more positive attitudes towards the subject and a stronger sense of their own mathematical identity. Although the detail is light here, it seemed in short that the experiences of the students in the non-traditional classrooms were akin to the mathematical practices of research mathematicians outlined above.

The Dance of Agency

The claims of Burton and the classroom evidence of Boaler (2003) together seem to make a strong case for considering the learning of mathematics to be like “working as a mathematician”. Conceptually, this requires engaging in what Pickering (1995) calls a “dance of agency”. In studying the practices of research scientist and mathematicians he noted that they choreographed a complex routine where at times they drew on their own agency as scientists or mathematicians, and yet at other times they would concede authority to the agency of their discipline and associated community of practice. This is like the interplay between the activity of mathematics and the content knowledge of mathematics that was highlighted earlier and rather than seeing the practice or knowledge-base being supreme, it reveals dialectic interdependence where the mathematician (at any level) requires both to meaningfully and to successfully engage in the mathematical enterprise. Likewise, teachers too need to engage in a dance of agency where they appraise and decide when to encourage and support the students’ own agency as mathematicians and when to defer to the authority of the disciple (e.g., the requirement to follow a standard procedure or form of presentation). It worth noting that mathematicians do defer to the agency of the discipline in their practice and it is this authority that is credible in a mathematics classroom. However, in traditional mathematics classrooms the authority usually resides

with the textbook and the teacher, both of which are temporary aspects of students' mathematical development and they do not endure as the discipline itself does.

Boaler's use of the dance of agency in her recent work (Boaler, 2003) illustrates the importance of learning having a robust and empowering identity in relation to mathematics. Knowing how and when to draw on mathematical ideas to solve problems is a critical part of the dance of agency. Boaler used examples of learners who could not solve tasks but drew on a range of skills, knowledge, and collective wisdom in order to solve such problems. This process is akin to that identified in Burton's work with research mathematicians. The practices offered by Boaler and Burton may offer a way forward and out of the quagmire of contemporary school mathematics that is being identified by many both inside and outside of education.

In the remainder of this paper, we draw on an example taken from a professional development that one of us undertook with a group of primary school teachers. We argue that the level of the learners is not the feature of the analysis as we contend this example can be used across all sectors of learning – primary, secondary, and preservice/in-service education. Rather, the analysis focuses on the ways of working that are the significant aspects of the example. These provide an illustration of how learners, in this case teachers, can draw on previous knowledge to work collectively to achieve a common goal. Collectively the goal is attained but not without considerable input from the learners. The input varies in form and timing, and helps to illustrate the powerful learning made possible when working in ways similar to mathematicians but also having a sense of agency that allows for the legitimate use of learners' understandings that enable the building of deeper understandings. However, as Boaler's work has highlighted, such success is dependent on the learners' sense of identity with mathematics and their sense of agency through which they can “dance” between the known and the unknown in order to build deeper understandings. It is for this reason we have used this example. After describing and illustrating the mathematical practices of these teachers, we draw on their example to discuss the features of mathematical classrooms that promote the development of robust mathematical identities through an authentic “dance of agency”. We use this illustrative example to show how the mathematical identity of learners may be constituted through particular practices of mathematics.

The data provided in the following example are drawn from field notes from the professional development activity. The quotes and drawings are those written by the observer and are representative of the discussion made by the participants as no formal recording tools (tape recorders) were used. The data were triangulated with participants so that they are an accurate summation of the interactions in the workshop.

Sum of the Interior Angles of an Octagon: A Working Example

A group of primary school teachers have been working on problems as part of a professional development activity. A standard geometry task is provided where they have to work out the sum of the interior angles of an octagon. There is some discussion as to what an octagon was, and how many sides it had. Once this is clarified, the teachers work in small groups.

I have no idea on how to work this out.

Well if you look at it you can divide it into triangles. See, there are 8 triangles. Each triangle has got 180° so to work out what the angles are on the bottom of the triangle, you have to work out how many degrees are in the top angle there [draws an arrow to the centre, see Figure 1].

Ah, so that is 360° divided by 8.

Huh?

Well you know that there are 360° in a circle [draws a circle around the centre] and you can see there are 8 triangles making up that circle.

So, $360 \div 8$ is [some talk on how to work this out, two teachers use pencil and paper for the division] ... 45.

OK now what we have to do is work out how big the other angles are. They are the same size so you take 45 from 180 and then divide by 2.

Why?

Well there are two angles [points to the two angles at the bottom of one triangle] and we need to see how big one is.

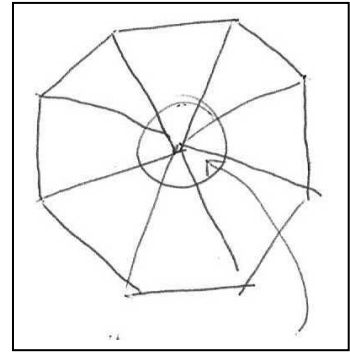


Figure 1. Octagon and eight triangles.

The discussion continues so that the group identify the size of one of the interior angles of the constructed triangles as being 67.5

There is some discussion that it cannot be right as the leader would not have given them an angle with a half in it. Calculations are checked and the answer is seen to be correct. Someone then suggests that they have to multiply it by 8 so it would not be a “half number” any more. Someone else in the group comments that it can not be right as the number they have calculated is less than 90° which would make for a less than “straight angle” [assumed to mean a “right angle”]. There is some discussion and movement of the shape and then agreement that they have done something wrong.

I know what it is... that is only half of the angle. See look, we have worked out half of the angle, the other part is in the triangle next door.

You’re right, so the size of one angle is really double what we found so that makes it 135. And that is bigger than 90 so we must be right now.

Ok, then we multiply by 8 and find out what the total size is.

Someone in the group then multiplies 135 by 8 using a pencil-and-paper method to come to an answer of 1080.

Once the group has finished, the leader then asks them to find out what it might be for a hexagon and some other shapes. The group goes through a similar process, this time drawing the hexagon, finding the magnitude of the central angle and then the size of each interior base angle. This is then doubled and multiplied by 6. At this point, a woman who has not contributed to much of the discussion interrupts and poses the following:

You know what we are doing... making more work for ourselves. Look at this. You divided the 120 by 2 and got the size of the angle inside the triangle and then you doubled it. We halved and then doubled so we have just done the same thing twice.

The teachers then go on to do two more shapes of their own choosing. The leader then poses the problem to see if they can make a prediction for any shape and how would they do it. The response is that this means they need to make a formula for the problem.

Group one made a table for their results. (Figure 2.) Aside from the triangle which they knew had 180, they had only made shapes with even numbers of sides so that it looked like:

Hey, look at that you can see a pattern there. Each time we go up by 2 sides, it gets bigger by 360. That is a square so if we only increased by one side it would be getting bigger by 180° – that is a triangle.

However, this group was unable to move beyond this observation to make a more generalisable statement.

Group two used a similar method and when it came to the discussion at the end of the session where groups shared their findings, this group explained that they found that the pattern was “increasing by 180° each time a side was added to a shape” but you could not go below 1 triangle as this was the lowest point. One teacher explained the generalisation as follows:

We found that what the pattern is that each shape is the number of sides takeaway 2 and then you multiply by 180° . So if you use a hexagon as the example, you can see that it has 6 sides but if you takeaway 2, you have 4 and then if you multiply it by 180 you get the sum of the interior angles. We thought you could say it like (number of sides minus 2) and then multiply by 180 so that is $(n-2) \times 180$. We checked it out with the others and it worked. So if you use the triangle. It has 3 sides, so that is $3-1$ and then times 180 so that is 180 and that is right.

sides	$^\circ$	Δs
3	180	1
4	360	2
5		
6	720	4
7		
8	1080	6
9		
10	1440	8

Figure 2. Table of sides, angles, and number of triangles.

Coming to Understand “Working as a Mathematician”

In drawing on Burton’s and Boaler’s work, we propose that there are three elements to developing a sense of working as a mathematician (see Figure 3). There are the cognitive aspects of knowing mathematics and thinking like a mathematician. Burton draws considerably on the cognitive features of working mathematically. Both Boaler and Burton recognise the importance of the social context within which learning occurs. Railside’s community has been strongly influenced by Complex Instruction (Cohen & Lotan, 1997; Cohen, Lotan, Scarloss, & Arellano, 1999) in terms of organising the learning environment. Burton draws more closely on the communities of practice literature (Wenger, 1998) to theorise her position and where she sees that “knowledge and the knower are mutually constituted within these dialogic communities” (1999, p. 132). Collectively the two positions provide a more comprehensive picture of the potential for classroom practice. Finally, the focus of both authors, and this paper, is that of mathematics.

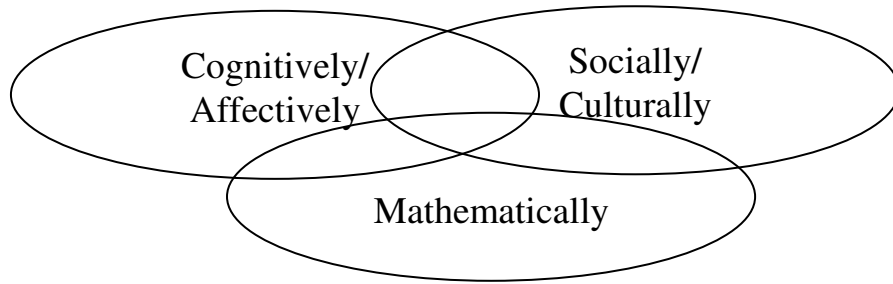


Figure 3. Aspects of working as a mathematician.

What can be seen in this example are a number of features about working as a mathematician. We take from the example used to illustrate aspects of these three constructs related to the notion of working as a mathematician and the importance of agency in this process.

Socially

For us, we define the context within which learning and working is occurring as the social dimension. This includes the ways in which the learning environment is organised along with the social and cultural dispositions that learners bring to that environment. From this example, we can see a number of features that enable learners to work as mathematicians.

Group work. Being part of a group and working as a collective enabled the teachers to share their knowledge, which is often tacit and not well understood. Drawing on this example, the teachers did not know the formula and so relied on bringing their collective wisdom enabled them to fill in gaps in each other's knowledge.

Collaborative talk. The interactions between the participants were focused on the task and enabled them to talk through observations. Having some participants working on the task and other observing enabled the observers to gain insights into the actions. In this case, one of the teachers was able to “see” that her colleagues were halving and then doubling. Being able to provide this input in a non-threatening way to colleagues enabled the group to move forward.

Ethos. The environment established in this session was non-threatening and supportive so that learners could actively engage in the active at levels that met their current needs and understandings. This ethos has been documented in Boaler's studies (Boaler, 2002a, 2002b) as being one that enables learners to participate without threat and hence open up opportunities for participation and learning.

Agency. Participants were able to draw on their own understandings to the situation and use these to develop richer understandings that are strongly mathematical. Being able to draw on existing knowledge to solve the problem in non-traditional ways, enabled the task to be completed but also to allow the participants to gain a strong sense of achievement.

Task. The design of the task may be seen as quite traditional but the leader deviated from those practices often found in classrooms where rote procedures are applied to a range of questions and little opportunity is provided to develop richer understandings.

Extrapolating the task to find the generalisation enabled the teachers to develop ways of thinking mathematically and to construct their own formula/generalisation.

Working as a mathematician. This aspect of the learning environment is very different from the traditional classroom where the format is often as a “consumer” or user of mathematics so that mathematics is the end product rather than the product.

Mathematically

This aspect of working as a mathematician draws on features that can be considered as part of the mathematical content knowledge or the pedagogical content knowledge identified by Shulman (1986). These features are often distinctly mathematical and are what can be seen to differentiate mathematics from other curriculum areas. Unlike traditional classrooms where there is feature of rote-and-drill learning, textbook-based exercises and strong teacher direction, mathematicians employ practices that are quite different from school mathematics practices. Some of these are identified in this example.

Identifying patterns. Creating the table enabled the participants to observe a pattern. For some participants, they were only able to describe the pattern but not the generalisation.

Constructing generalisations. Part of working as a mathematician is about making the generalisable statement. In this case, the development of a formula for the interior angles of a 2-dimensional geometric shape was part of the task. Unlike traditional mathematics classrooms where the generalisation (i.e. the rule) is often the starting point and learners are encouraged to practice on examples, this learning enabled the participants to generate their own generalisation.

Using a simple example to test the hypothesis. Once a potential generalisation had been developed, the participants applied this to a simple example (the triangle) to check its validity. In this case, it worked so it appeared to the participants that the generalisation was valid. They also applied the generalisation to the examples that they had worked out (and recorded in the table) to check that the generalisation was valid in other examples.

Identifying Limits. As noted by one group, the limit in this activity was that the shape had to have more than three or more sides if the generalisation were to work.

Cognitively

Drawing from Burton’s work are aspects of cognition and other features of the internal features of working as a mathematician. We have identified particular features of cognition and dispositions that are part of the learners’ ways of approaching the tasks.

Thinking styles. Drawing on a range of thinking styles identified by Burton (2001) – visual, analytic and conceptual – we can see how most of the learners used a composite of these styles. From the example used, we can see that the learners engaged using a range of thinking styles which include verbalisation, drawing illustrations, and the use of tables to arrive at insights about the problem, the mathematics, and ways to solve the problem.

Insight/Intuition. Burton’s (2001) mathematicians referred to the “light being switched on”, which enabled them to see what works and what does not work without being overtly aware of how they gained such insights.

Making connections. What can be seen from this example is that various elements of mathematics have been linked together to form a coherent whole. Burton argues that it is akin to fitting the pieces of the jigsaw together (Burton, 2001). What can be seen in this

example is how the teachers have drawn on various aspects of mathematical knowledge, in particular their knowledge of triangles, to pool this knowledge in order to come up with a deeper appreciation of mathematical understanding.

Identity and the Dance of Agency

What becomes possible to see through this example is that the learning situation draws considerably on those aspects of working as a mathematician as identified by Burton's work and on the aspects of classrooms and teaching identified by Boaler's work. Boaler's work has been particularly powerful in illustrating the importance of agency and identity. When we consider the activity identified in this paper, we recognise that the three features – social, mathematical, and cognitive – are critical variables in the provision of quality learning opportunities. If we are to emerge from the current demise in mathematics education as identified at the start of this paper, then reforms are needed to enable change from the current, traditional practices to ones that are more empowering for learners. This requires not only a shift in pedagogy and curriculum but also in the dispositions of learners.- As noted by Zevenbergen (2005) many of the current practices in school mathematics create particular mathematical habitus which are far from empowering for learners and indeed encourage disengagement with the discipline. This example and our analysis of that practice highlight some of the features that foster the characteristics of working as a mathematician that have been identified through the combined work of Burton and Boaler. However, in this final section, we want draw more constructively on Boaler's notion of dance of agency. For her, this construct is critical as it enables the learners to draw on their mathematical understandings, to build on what they know, to construct deeper understandings. This is one of the fundamental premises of much mathematical learning but which is not that possible in many of mainstream classrooms due to the pedagogies being implemented. As shown in the Queensland School Longitudinal Reform Study (Education Queensland, 2001), the teaching of mathematics in schools is one of the most poorly taught areas of school curriculum and dominated by shallow teaching approaches with little scope for students to engage substantially with ideas and deep learning. The example here provides some insights into the ways in which a commonly used activity can be adjusted to allow for depth of learning. However, as Boaler's work highlights, learners must feel some sense of agency to be confident to draw on other forms of knowing in order to solve problems.

We contend that traditional classrooms would have fostered learning activities around the application of a formula for calculating the sum of interior angles. In this example, the participants could not remember this formula (and it was not provided) so they need to rely on their existing knowledge, the collective wisdom of the group and a sense that they could solve the problem. This sense of agency – where they could rely not only on their own knowledge in a legitimate sense, but also on the collective knowledge across the group – enabled them to gain a sense of learning and achievement through the completion of the task. We contend that such practice is far more enabling and develops a strong sense of agency and identity with mathematics.

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