Studies in the Zone of Proximal Awareness

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I have long promoted the conjecture that expressing generality lies at the heart of school algebra. Indeed, I have gone further to suggest that "a lesson without the opportunity for learners to generalise is not a mathematics lesson". It seems beyond doubt that experiencing and expressing generality is natural to human beings. The pedagogic issue is why there is so much resistance amongst teachers and learners to using this power in mathematics lessons. The notion of generalisation here includes both abstraction from context and generalisation within context. Pondering this question has led us to wonder why generalisation happens sometimes and not others, what can be done to prompt useful mathematical generalisation, and under what sorts of circumstances: in short, what are the conditions for and evidence of imminent or proximal generalisation?

The present paper arises from reflections on lessons involving the expression of generality, sometimes by learners and sometimes by teachers. Our reflections led us to try to organise and inter-relate a variety of forms of mathematical generalisation: empirical, structural, and generic; syntactic and semantic; metonymic and metaphoric; enactive, affective, and cognitive. The idea is to prepare the ground for further studies. Our reflections also led to the notion of a *Zone of Proximal Generalisation* as a particular case of a *Zone of Proximal Awareness*, in order to try to describe and distinguish differences in the imminence of appreciation and competent expression of generality in and by different learners at different times. This in turn opened up a domain of further investigation into various proximal zones as projections of Vygotsky's original intention with the *zone of proximal development* into the three classic dimensions of the psyche.

Introduction

Historically, algebra is usually seen as arising through a desire to be able to solve problems involving some unknown number or numbers. As Mary Boole (Tahta, 1972) put it, by "acknowledging your ignorance" you can denote what you do not know with a letter, and then manipulate that letter as if it were a number in order to express relationships and constraints arising from the problem. Support for this view can be found in the use in early algebra texts of the term *cosse* ("thing") as the "as-yet-unknown".

At the same time however, there is a pervasive historical thread by authors wanting to solve every problem, or trying to indicate that the solution to a particular problem is to be seen generically as a method for solving a whole class of similar problems (Gillings, 1972; Cardano, 1545; Viete, 1581). Authors used a variety of means for informing the reader of the "general rule", in words, and through the use of examples. Newton (1683) may have been one of the first to use letters to denote as-yet-unspecified parameters so as to solve a problem "in general".

There is however a conceptual commonality between the use of a letter to stand for an as-yet-unknown and the use of a letter to stand for an as-yet-unspecified parameter: both depend on the person to be stressing the letter as label rather than as the value of the label.

This means treating the letter as a slot (we now call it a variable) without attending to its contents, trusting that the contents will look after themselves. Flexible movement between attending to the label and attending to the content (syntax and semantics of expressions) is the essence of working effectively with expressions of generality.

Every learner who arrives at school walking and talking has displayed the power to perceive and express generality and the case has been argued in one form or another by many (Whitehead, 1932; Gattegno, 1970). The issue for teachers is therefore not whether learners are capable, nor even whether learners will use those powers in lessons, but how to foster and support the use of those powers in mathematical ways, not only within mathematics but also so as to use mathematics to make sense of the world.

Expressing Generality

It is worth mentioning in passing that stressing *expressing generality* as a root of and route to algebra (Mason, Graham, Pimm, & Gowar, 1984; Mason, 1996; Mason, Johnston-Wilder, & Graham, 2005) does not mean that working on patterns in number sequences and matchstick pictures is either necessary, or sufficient. A much more comprehensive view is implied, as articulated in the abstract.

Every mathematics lesson involves generalisation, and the sooner and more frequently that learners are invited to try to express those generalities through actions and then words, the more likely they are to appreciate what algebra can do for them. This corresponds closely with Vygotsky's notion that teaching converts ability to do something into ability to do something knowingly, that is, to "transform an ability 'in itself' into an ability 'for himself'" (van der Veer & Valsiner, 1991, p. 334), and to Gattegno's notion of schooling as educating "awareness of awareness" (Gattegno, 1987; see also Mason, 1998).

Kieran (2004) refers to work with patterned sequences in which learners express generalisations as *generational* aspects of algebra, contrasted with manipulation and use (Kieran's *transformational* and *global/meta*). Without the generational there is no purpose in the transformational. Furthermore, both the transformational and the global/meta emerge directly out of multiple and rich experiences of expressing generality as the range and sophistication of expressing generality merges with core ubiquitous mathematical themes such as freedom and constraint and doing and undoing (Mason et al., 2005).

The aim of this paper is to try to discern what might be going on for learners who are on the edge of expressing generality in some situation. In particular, what might be the effects on different learners of being in the presence of expressed generality, and what can be done as a teacher to try to maximise the effectiveness of enculturation into perceiving and expressing generality.

Generality in this paper includes both the usual uses such as recognition of a feature in a situation as a parameter that could be varied, the abstraction that takes place when people focus attention on their actions rather than on the objects on which they are acting, and mathematical abstraction in which context is back-grounded and structural relationships are put forward as properties that are treated as axioms.

For example, adopting the practice of counting-on, when guided, can lead, through the natural process of reflective abstraction (in Piagetian terms) to focus on the action, creating counting-on as a new awareness (Mitchelmore & White, 2004). This in turn can lead to a change of level through interiorisation of a higher psychological practice (in Vygotskian terms).

Ways of Working

My way of working starts with recent experience reminding me of past experience. It involves identifying phenomena and issues that seem to deserve elucidation so as to inform my future action and that of others. I seek accounts (in my own and in other people's data) that highlight or illustrate phenomena and issues, and I test these for resonance with colleagues. I also construct tasks through which it is possible to have a taste of what it might be like for people in similar situations to those observed. Sometimes the mathematical sophistication of the task is appropriate for learners, and sometimes appropriate for colleagues with more extensive mathematical experience. Then I try out those tasks on others, modifying them and honing them so that most people report not only recognition of the phenomena, but recognition of the distinctions that have proved fruitful, linked with possible actions to take when working with learners. I do not try to prove that specific strategies are guaranteed to produce specified results, either statistically or through observational data. I seek educated awareness, not mechanical reproduction. For me what matters is awareness in the moment of planning and of teaching. This means what is being attended to, how it is being attended to, and what possibilities for action come to mind. I am content to offer experiences that might sensitise others to notice opportunities (through making pertinent distinctions) to act freshly and effectively in situations.

Phenomena

We begin with some characteristic phenomena that highlight a need to clarify and precise what is meant by generalisation in the mathematics classroom. Some situations are described that are likely to be recognisable in experience and some tasks are offered through which you may experience directly some significant aspects of generalisation. After each example there are comments which inform and illustrate distinctions.

Observed Phenomena

The story begins with a report by the second author of a repeated phenomenon in her classroom.

> ΦA: in one lesson, a learner asserted that "anything times zero is zero!". Her voice tones suggested surprise, as if it was a new thought. In actual fact she had uttered this same generality several lessons previously, with similar surprise.

Comment. The activity of the "maths fairy", which intervenes to wipe learners' memories (Houssart, 2001; 2004), is one way to account for this phenomenon. Does the learner really not recognise the same generality again, or is there an element of giving the teacher what the learner thinks is valued, namely conjectures and surprised voice tones?

As Rowland (1995) observes, learners may be very tentative in expressing half-formed thoughts and partially formulated ideas. He draws attention to the fragility of self-esteem and the use of linguistic hedges, presumably in order to distance the conjecturer from the conjecture in case it is seen as silly or wrong. He recommends creating a zone of conjectural neutrality, in which what is said is considered independent of the person who says it, and treated as something which may need to be modified or augmented. Establishing an atmosphere in which people are expected to and are supported in expressing half-formed thoughts makes a vital contribution to mathematical development, and particularly to expressing and appreciating generality (Mason, 1988, p. 9). The whole point of a conjecturing atmosphere is to overcome such sensibilities.

Since generalisations are being perceived and expressed, but apparently forgotten, expressing generality is not in itself a guarantee of "learning". It may however be a sign of cognitive development, of the use of powers that can be evoked or even called upon explicitly in future lessons.

Often generalisations are appropriate, if some what ambiguously expressed.

> ΦB: Asked what was 3 + 5, and then 5 + 3, and again, 2 + 6 and 6 + 2, Q (aged about 6) sat thoughtfully for a few moments and then said "anything plus anything is anything plus anything".

Comment. This illustrates a (possible) awareness of a generality, expressed spontaneously in response to attention being directed to a few facts. However the expression of that generality is highly ambiguous and reminiscent of "alge-babble" (Malara & Navarra, 2003) or "emergent algebra" (Ainley, 1999), as an example of attempts to express something without a firm grasp of the grammar and syntax used by others. It suggests a sensitivity to generalisation through the fact that attention drawn to a few "facts" made them exemplary of a more general fact. Because of the ambiguous multiple use of "anything", it is impossible to tell from the account whether the learner's attention was on commutativity, or on the fact that different pairs of numbers can add up to the same thing. Although there is a taste of the empirical, the learner's attention seems to be on structure.

Sometimes learners generalise inappropriately from partial or even incorrect data:

> ΦC: Learners, invited to generalise the following observations
> $3 + 4 + 5 = 3 \times 4$; $6 + 7 + 8 + 9 + 10 = 5 \times 8$; $7 + 8 + 9 + 10 + 11 + 12 + 13 = 7 \times 10$
> focus on the first expression and propose that $n + n{+}1 + n{+}2 = n\,(n + 1)$.

Comment. A task similar to this is mentioned in Rowland (2001), and the same thing has happened in other places, with the same results: someone looks only at the first statement and tries to generalise. It seems as though the person treats both the 3s the same without recognising the structural role of the second 3. Perhaps the statement is read simply as a succession of numbers with an equality sign, rather than as a structural statement about arithmetic. At the same time, the learner ignores the other proffered statements, perhaps because one is in the habit of dealing with one thing at a time, perhaps because one single statement occupies the attention fully, or perhaps because it all looks too difficult (which it might do if you were not attending to the structural detail of the statements but simply seeing them as strings of symbols). The *relational* thinking necessary to make structural sense has been studied in arithmetic by Molina (2007), (Molina, Castro, & Mason, 2007) and in equations using algebra by Alexandrou-Leonidou and Philippou (2007). It involves a subtle but important transformation of the structure of learners' attention (Mason, 2003).

> ΦD1: Teacher's account: I am talking to the whole class about the way in which they derived the equation of a circle with radius 2 and centre (3, 5). I have written the equations $\sqrt{(x-3)^2 + (y-5)^2} = 2$ and $(x-3)^2 + (y-5)^2 = 4$ on the board. I ask "where did the 3 the 4 and the 5 come from in this (the second) equation?" Trevor replies that the 4 is the diameter of the circle. (Bills & Rowland, 1999, p. 110)

> ΦD2: In a lesson on right-angled triangles, the first two examples were a 6–8–10 and a 5–12–13 triangle. A learner observed that the area and perimeter were (numerically) the same and conjectured that "this happens every time". (Bills & Rowland, 1999, p. 104)

Comment. In ΦD1 the learner has discerned a 4 but treated it as a structural 2 times a particular 2, rather than as a particular 2 squared. The teacher was attempting to shift students' attention from the process (of deriving the equation) to surface but structural relationships (for example that the term on the right hand side is the radius squared). But it seems that Trevor's attention was already on the surface relationships, and what is more, was triggered into seeing 4 as double two rather than as 2 squared. This has the flavour of a mixture of empirical and structural awareness at the mercy of metonymies.

ΦD2 is typical of what happens when examples have unintended and unanticipated features to which a learner attends. From the learner's perspective, some aspect was stressed (and so seen as a dimension of possible variation) while other aspects were ignored (and so seen, at least for the moment, as invariant). Fortunately the learner expressed the conjecture out loud; often there are relationships that are perceived implicitly, below the surface of awareness, but which pervade future thinking. Fischbein (1993) coined the term *figural concept* to describe the way in which unintended features inappropriately become part of the concept as constructed by the learner. In a conjecturing atmosphere, it is possible to praise the act of conjecturing while at the same time critiquing the conjecture itself. Faced with a conjecture that seems implausible, it is natural to seek a counter-example, or even to characterise all objects with the stated property. Zaskis and Lilejdahl (2002) point to several similar ways in which learners inappropriately generalise.

By contrast, some generalisations require considerable effort. Rowland (2001) reports on learners struggling to see how a process of reasoning used in a particular case in number theory could be applied to a general case involving any prime number. The indeterminate nature of the general was hard to pin down, because the reasoning seemed to require knowledge of the particular numbers. Consequently, some learners stuck with particulars. In a more elementary setting, learners unaccustomed to expressing generality and then asked to "say how to do it" very often revert to "well if you had …" and use a particular. Sometimes it is possible to get a flavour of how they are seeing through the particular to the general; sometimes it is difficult to tell whether their "method" is seen as particular, but able to be carried out in many different particular cases. In these situations, learners seem poised somewhere between experiencing the possibility of a generality, experiencing a generality but not being able to articulate it, and expressing a generality.

Considerations like these raise questions about how generalities expressed publicly in lessons contribute to or obstruct the perception and expression of generality by learners. It all came to head in the following incident in the second author's classroom.

> ΦE: In a lesson with 15 yr olds about rationals and irrationals, learners proposed $\sqrt{17}, \sqrt{5}, \sqrt{18}$ as examples of irrationals. The teacher then asked for a rational "one" and was given $\sqrt{9}$, then $\sqrt{16}$ and $\sqrt{4}$ She then said to the class "so all square roots of square numbers are rational".

Comment. The overall aim of the teacher was to work on the definitions of rational and irrational. It took 25 minutes of example construction to reach the point of formulating a definition, and 8 minutes to formulate definitions. There were other opportunities like the square-root example, for generalising along the way. The teacher had choices to make as each possibility came up: whether to engage the learners in expressing a generality for themselves, or whether, judging by the fluency of construction and the flow of examples, most learners appreciated the particular generality being exemplified at the moment. The teacher could also have chosen to ignore byway generalities altogether, or as an intermediate strategy, she could have suggested in passing that there was a generality to pursue some other time.

It is tempting for the teacher to utter "the generality" that she assumes everyone has experienced. But what is the effect of her statement on learners who have not yet become consciously aware of the property, or on learners who are vaguely aware but have not yet isolated it as a phenomenon and expressed it to themselves, or on learners for whom it is an almost unnecessary statement? For these latter, it may act as confirmation of what they were already well aware of, an important role for teachers to play. For learners who had not yet expressed it for themselves, it could act as a crystallisation, a bringing to the surface what they now recognised they had been aware of, albeit not explicitly. However, it could also serve to take away that moment of "things falling into place", when the act of generalisation releases just a little bit of energy in the form of pleasure or surprise as several particulars are subsumed under one label, as in "aha! that's what's going on!". For some learners the statement may simply pass them by because it does not speak to their current thinking, does not fit with, amplify, or summarise nascent awareness.

There is a direct analogy between the states being described here and the forms of noticing identified in Mason (2002): not noticing at all (below the surface of awarenesses that can be resonated); noticing but not marking (able to be resonated, in the sense that when someone else draws attention to it you recognise what is being described but you could not have initiated such a description yourself); marking (where you could initiate a description yourself); and recording (making some sort of external note about what is noticed).

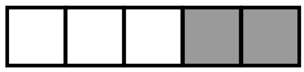
Before developing the notion of "being poised to generalise", here are some tasks which may provide an explicit taste of the same phenomena.

Experiential Phenomena

The following tasks may afford some opportunity to experience freshly some of the perceptions, states and issues arising from the previous phenomena.

Task A: Can You See …?

While gazing at the following diagram, can you "see"…



two-fifths of something? three fifths of something? two-thirds of something? one third of something? three-halves of something? five thirds of something? two thirds of three-halves of something? What other fraction calculations can you "see" directly? (Thompson, 2002)

Comment. The prompt to "try to see" signals a shift in how you attend to the figure, what you stress and consequently what you ignore, which Gattegno (1987) proposed as the mechanism of generalisation. What is available for generalisation is the particular fractional parts used, the particular diagram, and the way of working (stressing seeing rather than writing down).

Task B: Differing Products

Extend and generalise the following facts

$3\times 2 - 3 - 2 = 2\times 1 - 1$			
$4\times 3 - 4 - 3 = 3\times 2 - 1$	$4\times 2 - 4 - 2 = 3\times 1 - 1$		
$5\times 4 - 5 - 4 = 4\times 3 - 1$	$5\times 3 - 5 - 3 = 4\times 2 - 1$	$5\times 2 - 5 - 2 = 4\times 1 - 1$	
$6\times 5 - 6 - 5 = 5\times 4 - 1$	$6\times 4 - 6 - 4 = 5\times 3 - 1$	$6\times 3 - 6 - 3 = 5\times 2 - 1$	$6\times 2 - 6 - 2 = 5\times 1 - 1$

Comment. There are various ways to "see" structure. Watson (2000) used the metaphor of splitting wood to capture the natural propensity to recognise and pick up on a flow of familiarity, referring to it as "going with the grain": following or making use of familiar structure such as here with the flow of natural numbers. She uses the metaphor as a reminder that in order to make sense and to learn from the recognition of such patterns, it is necessary to "go across the grain", exposing the structure of the wood, and by analogy, the structure responsible for the perceived patterns. Going with the grain has been called recursive, or iterative, because it focuses on how the next term is obtained from previous terms. Going across the grain involves structural generalisation.

Here, going with the grain detects the flow of natural numbers both horizontally and vertically. Stressing what is invariant provides a skeleton with which to flesh out descriptions of what is changing, and how. Having more than one thing changing at a time often causes particular difficulties for people trying to generalise (Mason, 1996). Note that the presence of the equals sign emphasises the structural over the empirical, for it is the relationship that is being studied, not arrays of numbers from some unknown source.

Task C: Pólya Crosses Out

Write out the natural numbers in sequence for at least 10 terms. Now cross out the third, the sixth, the ninth, etc. and in a second line, record for each term the "sum so far" of the terms that are left. Repeat, but this time crossing out the second, fourth, sixth, … before forming the next line of "sums so far". You might recognise the final sequence.

1, 2, ~~3~~, 4, 5, ~~6~~, 7, 8, ~~9~~, 10, …   1, ~~3~~, 7, ~~12~~, 19, ~~27~~, 37, …   1, 8, 27, 64, …

Comment. This task appeared in Mason et al. (1982) but was taken from Pólya (1962). Many years later it showed up again in Conway and Guy (1996, pp. 63-65), who revealed its origins in Moessner (1952). The phenomenon of interest here is that apart from generalising to more rows and hence to bigger gaps in the first crossings-out, I was unable to see any other ways it might generalise. However, Conway and Guy displayed a whole world of intriguing and fascinatingly complex ways in which it generalises. For me the task is a reminder of the phenomenon that there are often dimensions of possible variation undiscovered in even the simplest of situations.

Questions Arising. These and other phenomena raise puzzling questions about why it is that, despite having displayed the power to generalise (and to particularise) in many different contexts, learners display a range of responses from no recall of previous generalisations (as in ΦA), through reticence, to downright refusal to make use of those powers in mathematics. What can teachers do to foster and sustain generalisation as a regular feature of mathematics lessons? It is sensible to start addressing this by distinguishing different forms of generalisation.

Forms of Generalisation

A distinction is often drawn, though mainly implicitly and with considerable variation by different authors, between *generalisation* and *abstraction*. This is related to a distinction between generalising the result of an action on objects as properties of the objects, and generalising (abstracting) that action away from the objects themselves. The abstracted action is then available to carry out on other (presumably similar) objects in other contexts. Then there is a distinction between *empirical*, also known as *generalisation from cases*, and *generic* (also sometimes referred to as *structural*) generalisation. It too is rather slippery as a distinction, and is employed differently by different authors. Distinctions are sometimes aimed at the mathematics, and sometimes at the activity of learners. For example, a distinction can be drawn between *syntactic* and *semantic* generalisation, and between *metonymic* and *metaphoric* generalisation, in an attempt to distinguish between the activity of someone satisfied with a *surface* approach to learning, and a *deep* approach. In a paper of this length there is not space to elaborate fully on all of these in detail, but it may be useful to attempt to relate them together as part of a web in which to try to catch a variety of generalisation experiences. Most generalisation, especially in algebra, is seen as a *cognitive* process, but sometimes generalisation happens without conscious awareness and thus could be considered as an *enactive* generalisation. The second author has observed that there is often an associated *affective* component of generalisation either supporting or inhibiting disposition to generalise mathematically in the future.

Elaboration. Piaget saw reflective abstraction as arising from a shift of attention from objects being acted upon, to the action itself. Thus mentally imagining linear transformations of the number line and the plane can involve calculating images of individual points and sets of points, but when attention shifts to the fact of the transformations themselves, then an algebra of transformations emerges. Some authors see this as *reification* of a process (Sfard, 1991), or as *proceptual* development (Gray & Tall, 1994).

Vygotsky stressed the need for ability “in itself” to be transformed into ability “for himself” (van der Veer & Valsiner, 1991 p. 331). This means shifting from carrying out an action, perhaps with considerable fluency but only when prompted or guided, to internalising it as an integral part of behavioural functioning, and so “knowing to” act in the moment, to use the ability “for oneself”. In a sense then, abstraction can be seen as a change of level, a shift in both the object and the structure of attention (what is attended to, and how).

Starting from an observation of the mathematician MacLane (1986), and Bills and Rowland (1999) studied *generalisation from cases* meaning generalisation that subsumes several particular cases such as in Task B, or as in seeing the expansion of $(a + b)^n$ as a generalisation of $(a + b)^2$, $(a + b)^3$ etc., which involves three parameters (a, b, and n) any of which can be seen as a variable. In this example the general rule summarises some features of the specific cases. It also asserts the plausibility of the generalisation in cases beyond those which have been examined, bridging the “epistemic gap” between known and unknown. They go on to quote a number of other mathematicians expressing similar sentiments. Generalisation from cases can take different forms:

- one case can be used generically to expose and articulate s*tructural generality*;
- a few cases can be used to reveal dimensions of possible variation;

- several cases can be used empirically or inductively/recursively.

ΦB, ΦC, ΦD2, and Task C illustrate aspects of empirical generalisation from cases. Sometimes however, it is possible to proceed directly from a single instance to the general simply by recognising one or more features that could be generalised (for example, as in Task A and ΦD1). *Generic* generalisation occurs when a single example is seen as generic, as illustrating relationships that are perceived as properties holding in a class of similar instances or examples. Balacheff (1988, p. 219) puts it clearly and elegantly:

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class.

The mathematician David Hilbert advocated a generic approach to difficult problems (Courant 1981, see also Mason & Pimm, 1984), and Krutetskii (1976, pp. 261-262) found high-achieving learners generalising through a single generic example. To see something generically is to look through the particulars to a generality, that is, to stress certain relationships and to treat them as properties common to all examples in the class being exemplified (Task A can be experienced this way). Task C is a reminder that seeing through a particular to a generality is no trivial matter. Watson and Mason (2005) refer to these particulars as *dimensions of possible variation* based on Marton's notion of *dimensions of variation* (Marton & Booth, 1997). The word *possible* was inserted because at any moment the teacher and the learners may be aware of different features that could be generalised. Furthermore, even when they are aware of the same dimension that could be varied, they may not be aware of the same *range of permissible* change of or by that variable. Typically in algebraic contexts it results in a parameter inserted for a constant.

Empirical generalisation always has some semantic content, in the sense that for the learner, surface relationships *are* the meaning, but they may not contact deep (mathematical) structure. ΦB offers an example of this. Task B is typical of tasks that can often be carried out with a surface rather than a deep approach, with the learner content to "get the answers" rather than to appreciate what is going on. Some intervention may be necessary by the teacher in order to provoke or promote a shift to appreciating structural relationships. (See later section for strategies.)

The terms *syntactic* and *semantic* generalisation provide another way to try to express the difference between what happens when learners use a predominantly *surface* or predominantly *deep* approach to learning (Marton & Säljö, 1976, 1984). Learners may be content with syntactic variation, a version of Watson's "going with the grain", which is evidenced most clearly when learners undertake a "copy and complete" type of exercise from a text by paying attention only to surface features. Again, in language stemming from grammar and linguistics, it may be possible to distinguish between *metonymic* generalisation that arises from surface associations triggered by personal and collective idiosyncrasies, homonyms, and the like (as in ΦD1), and *metaphoric* generalisation based on resonance with a sense of structure. The more experienced you are, the more structures you have encountered, and so the more likely you are to recognise an instance of a structure, not through calculated use of analogy, but through metaphoric resonance. Lakoff and Nunez (2000) argue that this can always be traced back to bodily sensation of some kind. Freudenthal (1983) advanced the case that a structure can be taught by locating a phenomenon that requires that structure in order to explain it.

The term *structural* generalisation applies to overlapping ground between empirical and generic generalisation, for the aim of empirical generalisation is (usually) to reveal structural properties that make the examples exemplary of some generality. Similarly, working with a generic example aims to locate structural invariance that is neither particular nor peculiar to the "example" being used. However, the real issue is that learners can act superficially to engage in empirical generalisation without reference to underpinning structure that generates the objects, or even structure within the objects themselves. Empirical is sometimes used to refer to "pattern spotting" in numbers without reference to what is generating those patterns. Without making use of the source of the "numbers" there can be no justification and hence no contact with underlying structure. ΦB, ΦC, and ΦD illustrate attempts and failures to contact structural generalisation. Hewitt (1992) highlights the natural tendency for teachers to try to short cut the lengthy process of learners using their own powers to generalise, by promoting a strategy of "draw up a table and look for a pattern in the numbers". Once reduced to mechanical behaviour, such activity becomes what he called "train spotting".

Pólya (1945) distinguished four states during empirical generalisation from cases: Observation of that particular case; Generalization; Conjecture formulation based on previous particular cases; and Conjecture verification with new particular cases (prior of course to justification of the generality). Cañadas and Castro (2007) find it useful to discriminate even more finely, inserting a further four: Organization of particular cases; Conjecture generalization; Search and prediction of patterns; and Justification of general conjecture. When observational data are organized in different ways, different patterns emerge, and cases can be organised more or less systematically or intentionally without prior detection of pattern. Cañadas and Castro describe "search for patterns" in terms of the learner seeking the "next" term in a sequence or in an array, without the pattern being thought of as applying to other cases. This is usefully seen as a structured form of attention in which the focus is on recognising local relationships in the specific situation, as distinct from perceiving properties that might apply in other cases (Mason, 2003; see also Pirie & Kieren, 1989; 1994). ΦD provides an illustration, and tasks B and C can be experienced in this way. When a relationship between terms is evident, learners may find their attention attracted to iterative relationships (often referred to as inductive or recursive) rather than to expressing the generality as a function of the position in the sequence or array. Stacey and MacGregor (1999) report considerable difficulty in provoking learners to shift from iterative/recursive generalisations to direct formulae, but then there may be little evident purpose or utility (Ainley, 1997) in making such a shift. A rich seam of mathematics has developed around methods of converting iterative/recursive generalisations into formulae (e.g., finite differences, formal power series).

Enactive generalisation is a description of situations in which the body perceives a generality before the intellect becomes aware of it, as when a learner shifts from copying each term in its entirety and starts repeating the invariant items before inserting the items that are changing. Most research has focused on *cognitive* generalisation, but the second author has observed indications that *affective* generalisation may accompany cognitive and enactive generalisation, in the form of altered dispositions either to engage, or not to engage in generalisation in the future.

What Generalisation is Like

How empirical generalisation comes about is obscured by the fact that even to see different objects as (potential) cases runs into the *exemplification paradox*: to come to see something as an example of something more general you already need to have a sense of that generality; to come to appreciate a generality you (usually) need to have some examples. One feature of learning to think and act mathematically is learning to cope with generality through particularising (in Pólya's language, specialising) in order to get a sense of underlying relationships, which when expressed as properties re-emerge as your own version of the original generality or an extension.

Gattegno suggested that generalisation (he included abstraction) comes about from *stressing* or fore-grounding some features and consequently *ignoring* or back-grounding others. This is manifested in the pedagogic strategy promulgated by Brown and Walter (1983) which they call *what if not*, in which some feature or aspect is interrogated for other possibilities. A particular version of this strategy is to read the statement out loud and to put special stress on one or another word. It is amazing how the stress attracts attention and invites asking why this word rather than some other word, and consideration of the possibility of changing it, of treating it as a dimension of possible variation.

The phrases *seeing the general through the particular* and *seeing the particular in the general* (Mason et al., 1985; see also Whitehead, 1911, pp. 4-5, 57) were formulated to try to capture that moment when a particular aspect is stressed and becomes the fore-ground of attention, so that other features fall away, and in that moment of stressing, other possibilities arise. It literally becomes a *dimension of possible variation,* almost as if a new world or new dimension opens up, however minor. There is often a corresponding sense of freedom, partly to do with subsuming previously different objects or actions under one heading, and partly to do with the freedom to choose different examples or instances. The person's example space is enriched (Watson & Mason, 2005). Sometimes the dimension was already part of the person's awareness, but the *range of permissible change* is extended, as in realising that the binomial theorem could apply to fractional as well as integer exponents, or more simply, that the *a* and the *b* could be not just integers, or fractions, or indeed any real number, but also algebraic expressions etc.

Although, as many have pointed out, generalisation is a power possessed by anyone who has learned to speak and to function in the material world, there are subtleties in evoking that power appropriately in mathematics classrooms. It is of course very tempting to try to do both the specialising and the generalising for learners: constructing tasks and suites of exercises that display particulars intended to be seen as instances, cases, or examples of a general class of such objects. It is then assumed that if learners "work through" the particular cases, they will emerge with a sense of the generalised whole. This assumption is contradicted by the observation that "one thing we do not seem to learn from experience, is that we rarely learn from experience alone". Something more is required.

That "something" is, presumably, what Piaget was trying to get at with his term *reflective abstraction*, what Vygotsky referred to as *internalisation of higher psychological functioning* through being in the presence of a relative expert displaying that functioning, and what Gattegno described as *integration through subordination*. Piaget and Gattegno stressed the natural and individual nature of such a transformation, of course supported and promoted by careful choices of teaching; Vygotsky stressed the importance of the social, including the incorporation of cultural tools and engaging in social interaction. Both dimensions are of course vital to a full appreciation.

Teachers' Influence

ΦE introduces the issue of how teacher intervention and articulation might influence learners' appreciation of generality. By uttering a generality herself, the teacher's utterance might be experienced by some learners as

- a crystallisation of a semi-focused awareness;

- a restating of the obvious;
- bridging or filling in an awareness of which they were not yet aware, and so taking away a generative experience; or
- nothing at all because it passes them by.

For some learners the utterance in ΦE might be part of the wallpaper of the lesson, for others it has a transformative action, and for others it confirms an awareness.

Just because some or several examples have been given by learners, and even when the flow of examples accelerates, awareness of the generality itself may not be present. It has to do with not just what learners are attending to, but precisely how they are attending.

These reflections on ΦD led the second author to the question of when it might be useful, effective, and appropriate for a teacher to utter a generality that generalises examples that learners have encountered or constructed for themselves. It was a short jump to the notion of a *Zone of Proximal Generality*. The idea was to describe and draw attention to various states of learner sensitivity to the possibility of generalisations in a particular setting, states in which learners are beginning to be aware of their subconscious awareness of a mathematical generalisation. For some learners, the generality was already present, perhaps explicitly, perhaps ready to be crystallised by someone expressing it in terms that linked to learners' experience. Such affirmation is often important for learners. For others however, the expression might displace and even block personal realisation that was underway and or imminent. But for some, any expression of that generality might have been ignored or even simply not heard: it was not in their current zone of proximal generality, perhaps because their attention was absorbed elsewhere.

The notion of a zone of proximal generality was soon recognised as a particular case of a *Zone of Proximal Awareness*. A generality is just one kind of awareness that can come to someone as a result of engaging in activity with cultural tools and using practices encouraged and displayed by a relative expert. The idea was to use the term to describe awarenesses that are imminent or available to learners, but which might not come to their attention or consciousness without specific interactions with mathematical tasks, cultural tools, colleagues, teacher, or some combination of these.

Since Vygotsky's original conception of the ZPD was as a dynamically emergent metaphoric space of possibilities describing potential development of conscious use of already familiar but all-engrossing behaviour, we recognised that his oft quoted definition has led researchers to a very truncated perception, a projection of the original idea into the behavioural aspects of the human psyche. What most people use is really a *zone of proximal behaviour*: what behaviour patterns might learners soon adopt for themselves?

Coining the term *zone of proximal generality* does more than provide a useful axis around which to accumulate classroom strategies and ways of analysing tasks. It also offers an opportunity to explore the implications of a zone of proximal awareness and a *zone of proximal affect*, which includes, for example, the *zone* of *proximal relevance* proposed by Mason and Watson (2005). It also links with *zones of promoted action* and *free movement* (Valsiner, 1988; Goos, 2004). Once conceived, the general notion of zones gave access to consideration of various zones in relation to the ZPD, which as a language might help us to articulate finer distinctions in the states and experiences of learners, including ourselves. These ideas require further study in order to elucidate their usefulness and interconnections.

Methodological Issues

There are tricky methodological issues however. It is tempting to describe learner behaviour in terms of "making a generalisation", when in fact all you have observed is behaviour consistent with making a generalisation. Learners may be immersed in action but only "going through the motions"; they may be immersed in action and reproducing behaviour patterns detected in the teacher and in peers, but without any underlying sense of meaning or larger story about why they are doing what they are doing. On the other hand, they may be immersed in action but generating or re-constructing that behaviour from themselves or with the support of peers. They may even be "acting for themselves", that is, "acting because", in the sense that they are able, when questioned, to articulate an appropriate mathematical justification.

ΦB underlines the teacher's predicament in seeking evidence that learners have appreciated, or integrated a generality into their functioning. When a teacher encounters a learner who is "acting as if" they are in possession of or aware of mathematical theorems and properties there is a dilemma:

- is the learner behaving as if he/she knows a theorem (or fact) without having explicit awareness of it?
- is the learner reproducing socially enculturated patterns of behaviour without being aware of them consciously, and/or without appreciating a larger picture and underlying principles?
- is the learner producing or reconstructing actions which are not only principled, but articulable at some level of sophistication without being prompted?

The first two may sometimes be distinguished by offering learners tasks in which the setting or situation is very different from those used to demonstrate or display the expected practices. A learner working from awareness, however subconsciously, may be able to use that awareness where someone working from socially induced behaviour will not. The third can arise spontaneously when one learner asks another for assistance or when there is some impetus to bring the reasoning to articulation.

It is very difficult to distinguish between "acting as if" and "acting because", or as Vygotsky put it, between a learner's quasi-concepts and true concepts (Confrey, 1994). The first situation was referred to by Vergnaud (1981) as *theorems-in-action*, to capture the sense of action that appears principled but which the learner may not be aware of explicitly, nor be able to articulate. The second and third are different responses of learners to what Brousseau (1984, 1997) called the *didactic contract*, from which arises the *didactic tension.* Since learners look to the teacher for the behaviour being sought, the more clearly the teacher indicates the behaviour being sought, the easier it is for learners to act "as if", to display that behaviour without actually generating it from themselves.

Distinguishing between the three responses is at best difficult, and at worst, delicate because excessive probing may have a negative rather than a positive impact. The act of probing for explicit description of the actions, for principles guiding those actions, or for justifications for those actions may have the effect of prompting the learner into a new level of awareness (and hence justify the use of the term Zone of Proximal Generalisation), but it may also have the effect of creating obstacles for the learner who may not recognise what is being asked, and so may develop affective, cognitive or even enactive blocks to further progress.

Pedagogic Strategies

How might a fistful of distinctions inform a teachers' future practice and so enhance the possibilities for learning? If distinctions remain at the intellectually academic level, so that teachers only know *about* them, then at best some ground may have been prepared. Where distinctions have become significant either because they help make sense of past experiences, or because they provide a label for sharpening noticing and a vocabulary for describing and analysing with colleagues, they begin to function, to inform choices of actions whether in planning or in the moment during a lesson. Distinctions become richer and more integral to a people's functioning when they are enriched by relevant personal experience. Significant choices involve choosing to act in some way that might not otherwise have come to mind, so that teachers find themselves "knowing to" act in the moment. Some strategies that have proved helpful in this respect regarding generalising include the following.

Guided Exploration. There are many difficulties with being a "guide on the side". Terms such as *scaffolding,* introduced by Wood et al. (1976), attempted to describe ways in which teachers could act as "consciousness for two" (Bruner, 1986, pp. 75-76) in supporting behaviour that would ultimately be taken up and directed by learners themselves, a process known as *fading* (Brown, Collins, & Duguid, 1989), through a process of progressively more and more indirect prompts until learners are adopting the behaviour spontaneously. The process is highly problematic, because a teacher acts in the moment; it is only later that the learners' behaviour makes it possible to describe the whole process as scaffolding and fading. Put another way, "teaching takes place in time; learning takes place over time".

What if Not (Stress and Ignore). This strategy was described in an earlier section. It involves stressing single words in an assertion, or some feature of an expression, and then asking what happens if that is allowed to change in some way.

Directing Attention (Stress and Ignore). More generally, learner attention can be directed towards something (as in Task A), with the consequence of back-grounding something else, but you cannot intentionally direct people to ignore something. Much of what teachers do in classrooms is to direct attention towards pertinent features. By directing attention in a structured manner it is possible to provoke awareness of sameness and difference, and so promote generalisation.

Watch What You Do. Encouraging learners not only to work on or construct an example but also to pay attention to how their bodies go about it, affords a perception that can often be translated into a generality. It can be used to enrich enactive generalisation, and to build links with structure (Mason et al., 2005).

Same and Different. Enculturating learners into the practice of looking for similarities and differences between objects under investigation leads quite directly to the perception and expression of salient features. Becoming aware of similarities and differences results in stressing or fore-grounding and consequently ignoring or back-grounding, which is the basis for both generalisation and abstraction. Brown and Coles (2000) report learners taking over the strategy and internalised it, integrating it into their everyday functioning in the mathematics classroom.

Say What You See. Getting learners to say something of what they see, and to listen to what others say they see often has the effect of re-directing learner attention, with the possibility of, at the very least, extending their awareness of possible interpretations or ways of seeing, and sometimes of detecting similarities that can emerge as generalities.

Predicting What is not Present. When learners are confronted with a possible pattern, it can help them express vaguely sensed generalities to try to predict other examples that are not present. They can then use *Tracking* to bring their awareness to expression.

Tracking Arithmetic. By deciding to forego closure on arithmetic operations concerning one number in a calculation, the progress of the value through the various steps in a calculation can be tracked. It is then a simple matter to ask what calculation would be needed if that tracked number changes: only its value changes, while the rest of the calculation remains the same. This is particularly useful in a difficult problem in which it is possible to check whether a proposed answer is correct, but hard to see how to find such a number. The arithmetic can be tracked on a "guess", the calculation generalised, and then an equation established whose solution(s) provide the desired answer. This is the method that Mary Boole called "acknowledging ignorance" (Tahta, 1972).

Why Does Generalisation Happen Sometimes and Not Others?

Changes in how people attend to something are not simple transformations. They involve a complex of experience, reflection, and perceived effectiveness (according to one's own criteria). To become robust they need integrated contributions from all three aspects of the psyche: cognition, enaction, and affect. Models or metaphors of learning that imply simple levels or steps to be ascended, or a few obstacles to be overcome, fail to account for the wide variation in learner experience and learner dispositions.

Teaching is not simply a matter of guiding or driving learners into appropriate patterns of behaviour, nor is it simply a matter of waiting for learners to display "readiness". Provoking generalisation is more about releasing learners' natural powers than it is about trying to force feed. Because promoting mathematical generalisation lies at the core of all mathematics teaching, at all ages, and because it concerns the development of higher psychological processes that are most likely to be accessible to learners if they are in the presence of someone more expert displaying disposition to and techniques for generalising, it is important for teachers to be seen to generalise, to value learners' attempts at generalisation, and to get out of learners' way so that they can generalise for themselves.

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References

Ainley, J. (1997). Constructing purpose in mathematical activity. In E. Pehkonen (Ed.), *Proceedings of the 21st annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.17-24). Lahti, Finland: PME.

Ainley, J. (1999). Doing algebra-type stuff: Emergent algebra in the primary school. In O. Zaslavsky (Ed.), *Proceedings of the 23rd annual conference of the International Group for the Psychology of Mathematics* (Vol. 2, pp. 9-16). Haifa, Israel: PME.

Alexandrou-Leonidou, V., & Philippou, G. (2007, February). *Elementary school students' understanding and use of the equal sign*. Paper presented at CERME 5 Working Group 6, Lanarka, Cyprus.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). London: Hodder and Stoughton.

Bills, L., & Rowland, T. (1999). Examples, generalisation and proof. In L. Brown (Ed.), *Making meanings in mathematics: Refereed proceedings of the British Society for Research into Learning Mathematics* (pp. 103-116). York: QED Publications.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. Steiner (Ed.), *Theory of mathematics education* (pp.110-119). Bielefeld: Institut fur Didaktik der Mathematik der Universitat Bielefeld.
Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactiques des mathématiques, 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.). Dordrecht: Kluwer.
Brown, L., & Coles, A. (2000). Same/different: a 'natural' way of learning mathematics. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, 153-160). Hiroshima, Japan: PME.
Brown, S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32-41.
Brown, S., & Walter, M. (1983). *The art of problem posing*. Philadelphia: Franklin Press.
Bruner, J. (1986). *Actual minds, possible worlds*, Cambridge: Harvard University Press.
Cañadas, M., & Castro, E. (2007). A proposal of categorization for analysing inductive reasoning. *Revista de Investigacíon en Didáctica de la Matemática*, 1(2), 67-78.
Cardano, G. (1545/1969). *Ars Magna or the rules of algebra* (T. Witmer, Trans.). New York: Dover.
Confrey, J. (1994). A theory of intellectual development. *For the Learning of Mathematics, 14*(3), 2-8.
Conway, J., & Guy, R. (1996). *The book of numbers*. New York: Copernicus, Springer-Verlag.
Courant, R. (1981). Reminiscences from Hilbert's Gottingen. *Mathematical Intelligencer*, *3*(4), 154-164.
Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139-162.
Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht: Reidel.
Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning*. London: Routledge & Kegan Paul.
Gattegno, C. (1987). *The science of education Part I: Theoretical considerations.* New York: Educational Solutions.
Gillings, R. (1972). *Mathematics in the time of the pharaohs*. New York: Dover.
Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education*, *35*, 258-285.
Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116-140.
Hewitt, D. (1992). Train spotters' paradise. *Mathematics Teaching*, *140*, 6-8.
Houssart, J. (2001). Rival classroom discourses and inquiry mathematics: The whisperers. *For The Learning of Mathematics*, *21*(3), 2–8.
Houssart, J. (2004). *Low attainers in primary mathematics: The whisperers and the maths fairy*. London: Routledge.
Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: The12th ICME Study* (pp. 21-33). Boston, MA: Kluwer..
Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children* (J.Teller, Trans.; J. Kilpatrick & I. Wirszup, Eds.). Chicago: University of Chicago Press.
Lakoff, G., & Nunez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.
MacLane, S. (1986). *Mathematics: Form and function*. New York: Springer-Verlag.
Malara, N. & Navarra, G. (2003) *ArAl Project. Arithmetic pathways towards favouring pre-algebraic thinking*. Bolgona: Pitagora.
Marton, F., & Booth, S. (1997). *Learning and awareness.* Hillsdale, NJ: Lawrence Erlbaum.
Marton, F., & Säljö, R. (1976). On qualitative differences in learning 2: Outcome as a function of the learner's conception of the task. *British Journal of Educational Psychology, 46*, 115-27.
Marton, F., & Saljo, R. (1984). Approaches to learning. In F. Marton, D. Hounsell, & N. Entwistle (Eds.), *The experience of learning* (pp. 36-55). Edinburgh: Scottish Academic Press.
Mason , J. (1988, 2nd revised edition 1998), *Learning & doing mathematics*. York: QED Books.
Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee(Eds.), *Approaches to algebra: Perspectives for research and teaching*. (pp. 65-86). Dordrecht: Kluwer.
Mason J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention *Journal of Mathematics Teacher Education*, *1*(3), 243-267.
Mason, J. (2002). *Researching Your own Practice: The Discipline of Noticing*, RoutledgeFalmer, London.
Mason, J. (2003). On the structure of attention in the learning of mathematics. *Australian Mathematics Teacher*, *59*(4), 17-25.

Mason, J., Graham, A., Pimm, D., & Gowar, N. (1985). *Routes to, roots of algebra.* Milton Keynes: The Open University.

Mason, J., Johnston-Wilder, S., & Graham, A. (2005). *Developing thinking in algebra*. London: Sage (Paul Chapman).

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277-290.

Mitchelmore, M., & White, P. (2004, July). *Teaching mathematical concepts: Instruction for abstraction*. Invited regular lecture at the International Congress on Mathematical Education, Copenhagen.

Moessner, A. (1952). Ein bemerkung über die potenzen der natürlichen zahlen. *S.–B. Math.-Nat. Kl. Bayer. Akad. Wiss., 29* (MR 14, 353b).

Molina, M. (2007). Desarrollo de pensiamento relacional y comprehensón del signo igual por alumnos de tercero de educación primaria. Unpublished doctoral dissertation, University of Granada, Granada.

Molina, M., Castro, E., & Mason, J. (2007, February). *Distinguishing approaches to solving true/false number sentences*. Paper presented at CERME 5, Working Group 6, Lanarka, Cyprus.

Newton, I. (1683). *The Mathematical Papers of Isaac Newton, Vol V* (D. Whiteside, Ed., 1964). Cambridge: Cambridge University Press.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it?, *Educational Studies in Mathematics*, *26*(2-3), 165-190.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics*, *9*(4), 7-11.

Pólya, G. (1945). *How to Solve it: a new aspect of mathematical method*. Princeton, USA: Princeton University Press.

Pólya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving*. New York: Wiley.

Rowland, T. (1995). Hedges in mathematical talk: linguistic pointers to uncertainty. *Educational Studies in Mathematics*, *29*(4), 327-353.

Rowland, T. (2001). Generic proofs in number theory. In S. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*. (pp. 157-184). Westport, CT: Ablex Publishing.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics,* 22, 1-36.

Stacey, K., & MacGregor, M. (1999). Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal*, *11*(1), 25-38.

Tahta, D. (1972). *A Boolean anthology: selected writings of Mary Boole on mathematics education/* Derby: Association of Teachers of Mathematics.

Thompson, P. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education* (pp. 191-212). Dordrecht: Kluwer.

Valsiner, J. (1988). *Developmental psychology in the Soviet Union*. Brighton: Harvester.

van der Veer, R., & Valsiner, J. (1991). *Understanding Vygotsky*. London: Blackwell.

Vergnaud, G. (1981). Quelques orientations théoriques et méthodologiques des recherches Françaises en didactique des mathématiques. *Actes du Vième Colloque de PME* (vol. 2 pp. 7-17). Grenoble: Edition IMAG.

Viète, F. (1591/1983). *The analytic art: Nine studies in algebra, geometry, and trigonometry from the Opus restitutae mathematicae analyseos, seu, Algebra nova / by Francois Viète* (T. Witmer, Trans.). Kent, OH: Kent State University Press.

Watson, A. (2000). Going across the grain: Mathematical generalisation in a group of low attainers. *Nordisk Matematikk Didaktikk, 8*(1), 7–22.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum.

Whitehead, A. (1911). *An introduction to mathematic*s. Oxford: Oxford University Press.

Whitehead, A. (1932). *The aims of education and other essays* London: Williams and Norgate.

Wood, P., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, *17,* 89-100.

Zaskis, R., & Liljedahl, P. (2002). Repeating patterns as a gateway. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 212-216). Norwich, UK: PME.