Cognitive and Metacognitive Aspects of Mathematical Problem Solving: An Emerging Model

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This paper presents results from a study of the non-routine mathematical problem solving employed by 17 preservice teachers. Analysis of task-based interviews led to the identification of five cognitive phases: engagement, transformation-formulation, implementation, evaluation, and internalisation. Corresponding metacognitive behaviours were associated with each of these cognitive phases. A five-phase model for problem-solving, which incorporates multiple pathways, is described. Since various pathways between the categories are possible, the model accommodates the range of metacognitive approaches used by students.

Introduction

Although systematic research on problem solving in mathematics began in the 1970s, it was only in the 1980s that more intensive research was undertaken (Lester, 1994). More recent emphases on problem-solving curriculum development and research in the United States could be attributed to significant publications of the National Council of Teachers of Mathematics (NCTM). In each of *An Agenda for Action* (NCTM, 1980), *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and *Principles and Standards for School Mathematics* (NCTM, 2000), the Council recommended that problem solving be the focus of school mathematics along with reasoning, connecting, communicating, and representing. Lester (1994) noted that, since publication of *An Agenda for Action*, problem solving is the most written about, but least understood area, of the mathematics curriculum.

In recent years, a shift in emphasis from *doing* a mathematical activity to *thinking* about the relationship between mathematical ideas has been evident (Adibina & Putt, 1998; Schoenfeld, 1985a; Trafton & Midgett, 2001). Metacognitive processes, according to Schoenfeld (1985b), include assessing one's own knowledge, formulating a plan of attack, selecting strategies, and monitoring and evaluating progress. Thus, metacognitive processes focus on students' ability monitor and regulate their own cognitive processes employed during problem solving (Artzt & Armour-Thomas, 1992; Schoenfeld, 1992).

Many scholars have argued that emphasis on cognition without a corresponding emphasis on metacognitive thinking renders a problem-solving endeavour incomplete (Artzt & Armour-Thomas, 1992; Berardi-Coleta, Dominowski, Buyer, & Rellinger, 1995; Kirkwood, 2000; Lin, 2001; Schoenfeld, 1985a). A rich store of knowledge is believed to be a necessary but not sufficient requirement for successful mathematical problem solving (Garofalo & Lester, 1985; Geiger & Galbraith, 1998; Schoenfeld, 1987; Silver, 1987). Although students may be equipped with knowledge or skills to interpret the statement of a problem, inefficient control mechanisms can be a major obstacle during solution attempts (Carlson,1999). Carlson concluded that, irrespective of the richness of students' knowledge bases, their inefficient control decisions often mean that known mathematical knowledge is not accessed, and general problem-solving strategies are therefore not employed.

Lin (1994) argued that a learner's internal metacognitive functioning provided the key

to successful learning under learner-control situations. An inability to provide accurate monitoring, reflection, evaluation, and adjustment of learning, hinders learning and is indicative of poor metacognitive skills on the part of the learner. Stillman and Galbraith (1998) found that providing students with opportunities to make metacognitive decisions did not ensure that such decisions would be made, or that these decisions would be appropriate. Stillman and Galbraith concluded that, although a rich knowledge of metacognitve strategies and the appropriate application of this knowledge developed over an extended period of time, both were prerequisites to productive decision making.

Several research studies have concluded that metacognitive processes improve problem-solving performance (Artzt & Armour-Thomas, 1992; Goos & Galbraith, 1996; Kramarski & Mevarech, 1997). Metacognition is also believed to help students develop confidence to attempt authentic tasks (Kramarski, Mevarech, & Arami, 2002), and to help students overcome obstacles that arise during the problem-solving process (Goos, 1997; Pugalee, 2001; Stillman & Galbraith, 1998).

Distinguishing between what is cognitive from what is metacognitive has been problematic (Garofalo & Lester, 1985; Goos & Galbraith, 1996). To help make the study of metacognition more systematic, frameworks for metacognitive processes associated with learners' problem-solving performances across a wide range of domains have been proposed. For example, Davidson, Deuser and Sternberg (1994) identified four metacognitive processes that may be applicable in any domain: identifying and defining a problem; mentally representing the problem; planning how to proceed; evaluating what you know about your performance. Schoenfeld (1985a) developed a four-stage model which involved resources, heuristics, control, and belief systems. Garofalo and Lester (1985) developed a cognitive-metacognitive framework that consisted of four categories: orientation, organisation, execution, and verification. The cognitive-metacognitive framework proposed by Artzt and Armour-Thomas (1992) consisted of eight categories: read, understand, analyse, explore, plan, implement, verify, and watch and listen. Geiger and Galbraith (1998) developed a script analysis framework that categorised metacognitive behaviours observed when students solved mathematical problems. Their framework included engagement, executive behaviours, resources, and beliefs. These models and frameworks, in fact, all used minor variations of Polya's (1957) four-stage model — understand, plan, carry out the plan, and look back.

Most studies reported in the literature have focused on the identification and classification of metacognitive strategies, and have used these classifications to suggest appropriate frameworks. Although such approaches are important in helping to develop a more detailed understanding of metacognition, and of possible relationships between cognition and metacognition, they fall short of providing a research base for helping students to develop monitoring and regulating behaviours during problem-solving activities. Such a research base is particularly important for preservice teachers who are developing problem-centred approaches in their teaching that will facilitate the development of metacognitive awareness and regulatory skills in their students.

This paper will report data from a study which focused on the range and patterns of metacognitive processes employed by a sample of preservice education students as they engaged in mathematical problem solving.

Research Design and Methodology

A qualitative research design involving clinical case studies that utilised structured, task-based clinical interviews was used as a primary means of data collection. Seventeen preservice teachers, who were enrolled in a problem-solving course in a large mid-western university, participated in this study. Data from two case studies will be summarised.

Since the tasks in this study were to be the main vehicles for generating data on metacognitive behaviours/actions and/or processes by individual problem solvers, care was taken in selecting appropriate mathematical tasks. Three criteria were used to select six problems for individual task-based interviews: (a) the problems selected needed to be within the students' reach, and should not require them to apply a mathematical concept or principle with which they were not familiar (Kroll, 1988); (b) the problems should be challenging; and (c) the problems should be non-routine — that is, the problems should not all require algebraic manipulations, nor should they be trivial. One of the problems, the egg-vendor problem, is shown in Figure 1.
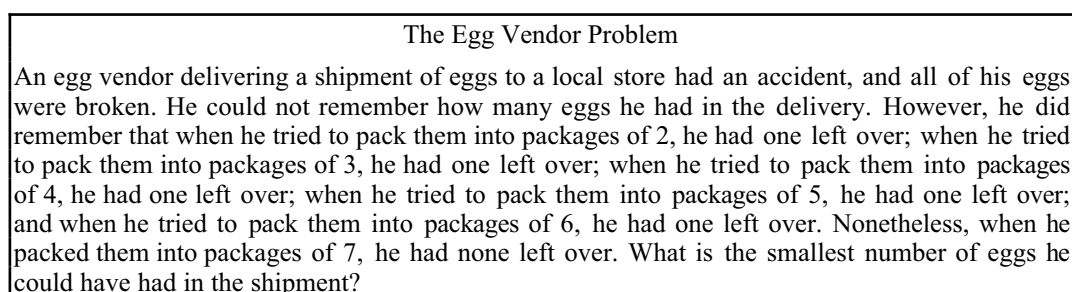

The Egg Vendor Problem

An egg vendor delivering a shipment of eggs to a local store had an accident, and all of his eggs were broken. He could not remember how many eggs he had in the delivery. However, he did remember that when he tried to pack them into packages of 2, he had one left over; when he tried to pack them into packages of 3, he had one left over; when he tried to pack them into packages of 4, he had one left over; when he tried to pack them into packages of 5, he had one left over; and when he tried to pack them into packages of 6, he had one left over. Nonetheless, when he packed them into packages of 7, he had none left over. What is the smallest number of eggs he could have had in the shipment?

Figure 1. The egg-vendor problem.

Results and Discussion

Task-Based Interviews

Transcripts from the individual, task-based interviews revealed different orientations and solution processes for each student. The solution processes used by different students working on the same problem showed strong individual characteristics, with differing degrees of understanding, depths of analysis, and control. Even the same student was found to exhibit different levels of sophistication and metacognitive behaviours across different problems. But despite these idiosyncratic characteristics, it was evident that certain patterns recurred in most individual's solution processes. These recurrent patterns became the stimulus for scrutinising transcripts carefully.

Figures 2 and 3 demonstrate how these phases can be described in terms of indicators which took place in the solution processes used by two of the participants, Trish and John.

A New Model to Describe the Problem-Solving Process

The constant comparative method (Maykut & Morehouse, 1994) was used in unitising and categorising behaviours. Five phases were identified: engagement, transformation-formulation, implementation, evaluation, and internalisation. Figure 4 gives details of the five-phase cognitive/metacognitive model which was formulated on the basis of the task-

based interview data. Possible cognitive and metacognitive behaviours are listed under each of the five phases. Task-based interview data in this study suggested that one or more (but not necessarily all) of the categories for each phase were evident for each problem-solving attempt by each problem solver.

Excerpts	Behaviour	Phase
(1) Read the problem 2) I think it in clear. It is asking for the least number the vendor should have. 3) The number has to be an odd number since it leaves a remainder of 1 when divided by 2.	- Initial engagement - Restatement - Analysis of the information	Engagement
4) I guess the number I am looking for should be of the form 7x+1 since it leaves 1 when packed as packages of 7 5) Oh! No! I read 'one' instead of 'none'. I have misread it completely. The plan I have as 7x=1 is wrong.	- Planning a solution strategy -Assessing consistency of the plan with conditions - Abandoned the plan	Transformation-Formulation
(6) I have to start all over again	- Reread the problem - Analyse the problem	Engagement
7) I am going to list all multiples of 7 and check the other conditions. I do not think it is a wise way but I will do it.	- Made a new plan - Assessed the plan for appropriateness	Transformation-Formulation
8) I do not know how long should I go. I have an idea. I think the number I am looking for is an odd number. So, I have to focus on odd multiples of 7.	- Implemented the plan. - Assessing the plan - Overwhelmed - Got a new idea - Terminated the Plan	Implementation
9) This time instead of adding 7 each time, I will start from 7 and add 14 each time to get odd multiples of 7	- Made a new plan - Devised how to carry out the new plan	Transformation-Formulation
10) I am stuck. It is the same thing. It is overwhelming. I do not think I am in the right track. But some of the things I observed are right though	- Implemented it - Assessed merits of plan - Got overwhelmed - Not confident on what she did - Terminated the plan	Implementation
11) Multiples of 5 and either in 0 or 5. Since the number I am looking for leaves a remainder of 1 when divided by 5, it should end either in 1 or 6. But since it has to be odd (because of 2), it should be a multiple of 7 that ends in 1.	- Reread the problem - Made a new observation - Analysed the new observation	Engagement
12) I think this should work. Let me list them and check the conditions	- Made a new plan - Assessed merits of new plan	Transformation-Formulation
13) Yes. It is 301. It has to be the smallest because my analysis is right and I did not leave out any number.	- Implemented the plan - Reached at a solution	Implementation
14) It is not really difficult but it requires a lot of analysis and observation.	- Reflected on difficulty level of the problem.	Internalisation

Figure 2. Excerpts and coding of Trish's solution to the egg-vendor problem.

Excerpts	Behaviour	Phase
1) Read the problem. 2) Although the context is about packing eggs, mathematically it asks to find a number which is a multiple of 7 and at the same time satisfy the other conditions .	- Initial engagement - Analysis of the information	Engagement
3) I'll model the problem algebraically. 4) Assigned x to represent the number of eggs the vendor had. 5) Argued that x – 1 was evenly divisible by 2, 3, 4, 5, and 6.	- Planning a solution strategy -Unsure how to proceed - Abandoned the plan	Transformation-Formulation
6) Re-read the problem.	- Re-analysed the problem - Reflected in silence	Engagement
7) I do not think I can handle the algebraic model. 8) It may call for Diophantine equations and I do not think I need such hard machinery here. 9) I, instead, want to look at it differently.	- Abandoned algebraic approach - Considered but rejected another approach - Tried to find a new strategy	Transformation-Formulation
10) Since there is 5, the number I am looking for (number of eggs) should end in either 1 or 6. 11) But the one ending in 6 is not good. 12) If it ends in 6, then, it would not leave a remainder of 1 when divided by two.	- Implemented a new plan - Assessed the plan - Reflected on appropriateness of actions	Implementation
13) This leaves me to look for multiples of 7 that end in 1. 14) Wrote x – 1 ends in 0 but did not use it in generating either some kind of equation or those numbers ending in 0. 15) Started generating multiples of 7 ending in 1 as 21, 91, 161 and checked other conditions. 16) I am not happy with what I am doing now. It is just guessing.	- Made a new plan - Re-attempted an algebraic plan - Explored feasibility of plan - Reflected on conditions - Rejected plan	Transformation-Formulation
17) I need to reread the problem again and see if I can come up with a different method.	- Re-read problem - Reflected on problem	Engagement
18) I am thinking whether it is related to 6! = 720..	- Made new conjecture	Transformation-Formulation
19) And 121 is a multiple of 7 and 720 satisfies all the conditions.	- Implemented the plan	Implementation
20) Is that the answer? 721 has to be the smallest number. 21) Decided that 721 would be answer even though he realised that he had no any idea whether it was the smallest or not.	- Reflected on solution - Made decision to accept solution	Evaluation
22) Well, the problem was not bad but I could not see the smartest way and I am not sure still whether I have answered it or not.	- Reflected on approaches used	Internalisation

Figure 3. Excerpts and coding of John's attempted solution to the egg-vendor problem.

Description of Paths

The problem-solving behaviours observed for Trish (Figure 2) and for John (Figure 3) do not represent a smooth path from one of the five phases to the next. The data from task-based interviews suggests that the paths problem solvers followed were mediated by

rereading of the problem. In other words, re-reading served as a catalyst for metacognitive decisions to take place either in the form of choosing a path or choosing other metacognitive actions within a specific cognitive phase in which problem solvers engaged. The engagement of problem solvers in controlling and regulating their actions in either selecting or abandoning a specific path corresponds to Casey's (1978) error analysis hierarchy. According to Casey, for example, a problem solver might decide to reread the problem in order to check that all relevant information has been taken into account.

It should also be noted that these paths may be cyclic as students engage in a series of assessments and other paths before they decide to engage in a different path. For example, Trish was engaged in a series of transformation-formulation-implementation phases before she decided to analyse the problem again (see Figure 2).

Possible paths between the five phases that can be taken by problem solvers are represented in the flow chart shown in Figure 5. Depending on the background of individuals, their understanding, and their ability to analyse the problem-solving situation, different paths are likely to be taken. The paths represent the consequences of possible decisions made by the problem solver as a result of metacognitive behaviours.
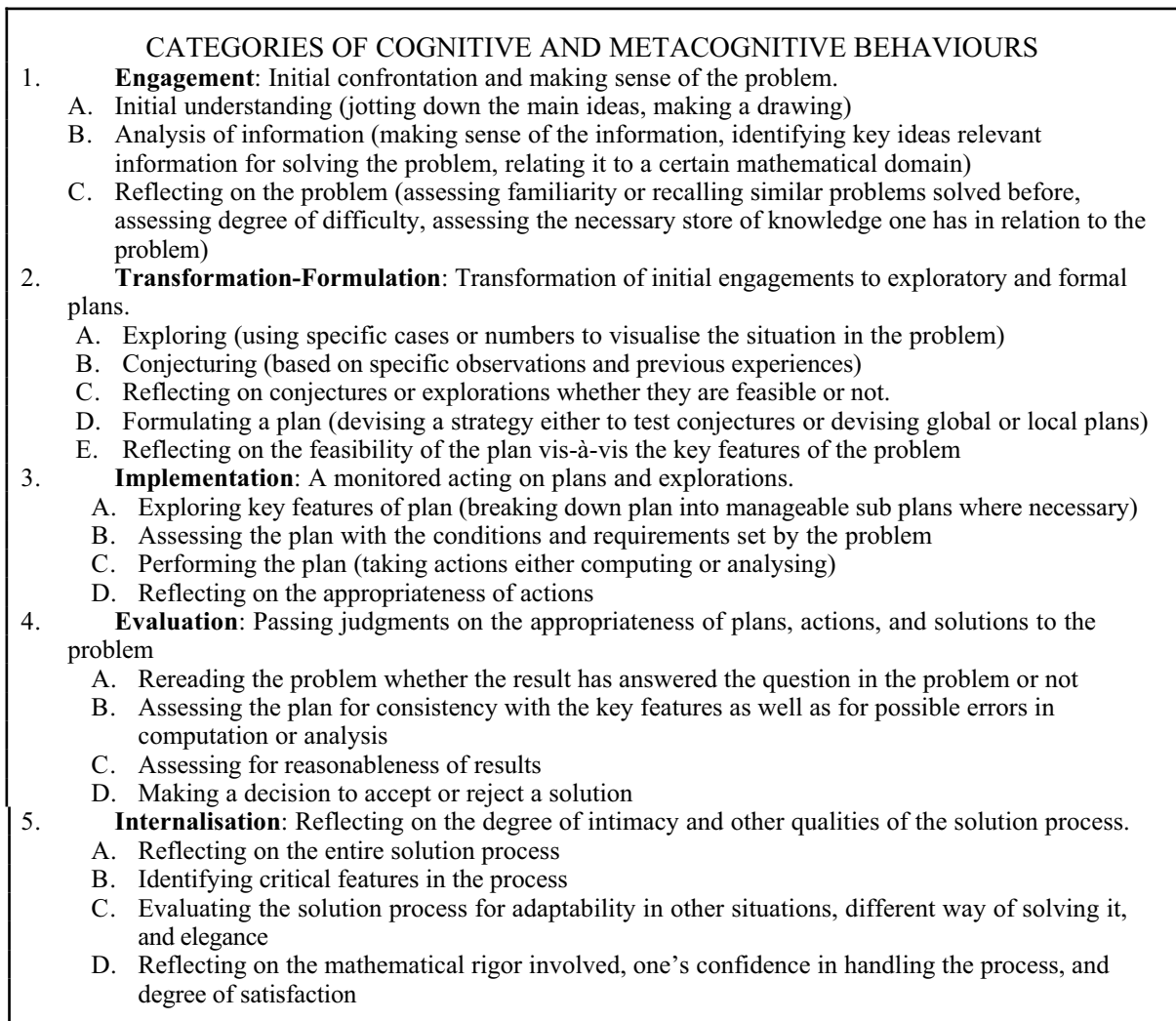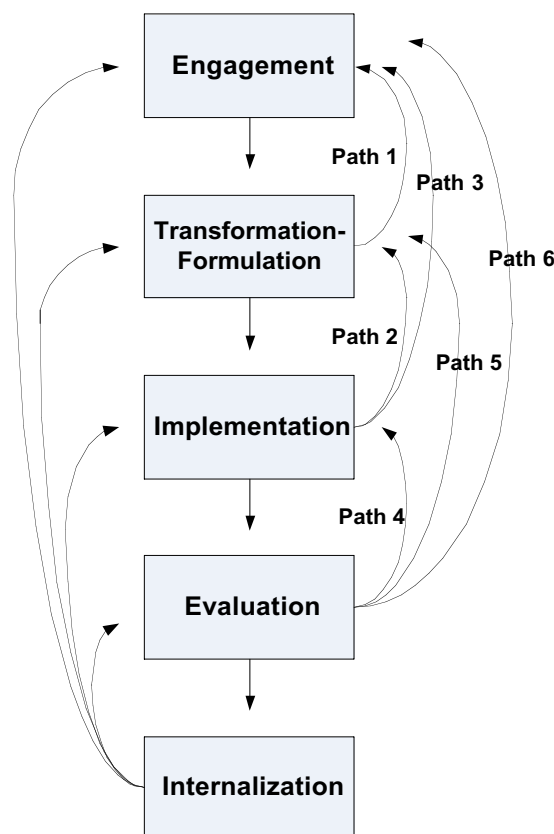
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CATEGORIES OF COGNITIVE AND METACOGNITIVE BEHAVIOURS

1. **Engagement**: Initial confrontation and making sense of the problem.
   A. Initial understanding (jotting down the main ideas, making a drawing)
   B. Analysis of information (making sense of the information, identifying key ideas relevant information for solving the problem, relating it to a certain mathematical domain)
   C. Reflecting on the problem (assessing familiarity or recalling similar problems solved before, assessing degree of difficulty, assessing the necessary store of knowledge one has in relation to the problem)
2. **Transformation-Formulation**: Transformation of initial engagements to exploratory and formal plans.
   A. Exploring (using specific cases or numbers to visualise the situation in the problem)
   B. Conjecturing (based on specific observations and previous experiences)
   C. Reflecting on conjectures or explorations whether they are feasible or not.
   D. Formulating a plan (devising a strategy either to test conjectures or devising global or local plans)
   E. Reflecting on the feasibility of the plan vis-à-vis the key features of the problem
3. **Implementation**: A monitored acting on plans and explorations.
   A. Exploring key features of plan (breaking down plan into manageable sub plans where necessary)
   B. Assessing the plan with the conditions and requirements set by the problem
   C. Performing the plan (taking actions either computing or analysing)
   D. Reflecting on the appropriateness of actions
4. **Evaluation**: Passing judgments on the appropriateness of plans, actions, and solutions to the problem
   A. Rereading the problem whether the result has answered the question in the problem or not
   B. Assessing the plan for consistency with the key features as well as for possible errors in computation or analysis
   C. Assessing for reasonableness of results
   D. Making a decision to accept or reject a solution
5. **Internalisation**: Reflecting on the degree of intimacy and other qualities of the solution process.
   A. Reflecting on the entire solution process
   B. Identifying critical features in the process
   C. Evaluating the solution process for adaptability in other situations, different way of solving it, and elegance
   D. Reflecting on the mathematical rigor involved, one's confidence in handling the process, and degree of satisfaction

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Figure 4. The problem-solving model.

Figure 5. Flow chart of the cognitive processes in the problem-solving model.

Conclusions

The five-phase model suggested by data from this study takes into account the symbiotic relationship between cognition and metacognition. Within each phase, indicators of the existence of metacognitive behaviours have been identified and described. Garofalo and Lester (1985) and Artzt and Armour-Thomas (1992), in using the term cognitive-metacognitive, recognised the inter-dependence of cognitive and metacognitive behaviours, although their models did not account for the complexity of the relationship.

The model proposed in this study has distinctive characteristics which differentiates it from others. First, reflection is an integral part of each category and of the entire model. Second, the last phase, internalisation is not present in other models as a separate phase. Internalisation shows the degree of intimacy the problem solver has with the process in general, and his or her inquiry for elegance and extension, in particular. In this phase, problem solvers reflect on the mathematical rigor of the problem, search for elegant solutions, express their satisfaction or dissatisfaction, and reflect on their confidence in handling similar problems.

The importance of nurturing metacognitive behaviours during every phase of mathematical problem solving is inherent in the flow chart (Figure 5). Without metacognitive monitoring, students are less likely to take one of the many paths available to them, and almost certainly are less likely to arrive at an elegant mathematical solution.

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