Student-Engineered 'Space to Think'

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The nature of creative and insightful thinking of Year 8 students in mathematics classes was studied through simultaneous examination of post-lesson video stimulated interviews and lesson video. One case is used to illustrate these findings. The concept of Space to Think emerged as a space manoeuvred by each of the five (out of eighty-six) students from Australia and the USA who creatively developed new knowledge. This Space to Think illuminates pedagogical moves that provide opportunities for creative thinking.

Introduction

'Discovering complexity' (creative mathematical thinking) has previously been identified as requiring the conditions for 'flow' during mathematical problem solving (Williams, 2002). Flow occurs when a person or group pursues a 'spontaneously' set challenge that requires the development of new skills (Csikszentmihalyi, & Csikszentmihalyi, 1992). Discovering complexity during mathematical problem solving involves spontaneously exploring a mathematical complexity (intellectual challenge) that was not evident at the commencement of the task (Williams, 2002). This activity produces new conceptual knowledge. The term 'spontaneous' refers to student-directed cognitive activity over a time interval when there is no mathematical input from external sources (Williams, 2004). This paper explores the nature of student activity during creative mathematical thinking and the 'social' and 'personal' influences upon it. One case is used to illustrate the findings.

Theoretical Framework

Personal and social factors can influence the development of new conceptual understanding ('abstracting') (Dreyfus, Hershkowitz, & Schwarz, 2001). Personal factors influencing this process include 'mathematical background' and 'inclination to explore'. Social factors that contribute to the development of new knowledge are examined through six 'social elements of the process of abstracting' (Dreyfus, Hershkowitz, & Schwarz, 2001). The mathematical background of the student influences the 'cognitive artefacts' they can 'recognize' in developing appropriate mathematical procedures to solve problems. Dreyfus, Hershkowitz, and Schwarz (2001) identified three epistemic actions ('recognizing', 'building-with', and 'constructing') occurring during the process of abstracting. Recognizing involves identifying appropriate cognitive artefacts that can contribute to building-with and constructing. Building-with involves using previously known mathematics (cognitive artefacts), and constructing is the process of creating a new mathematical structure or new insight. This study illuminates the meaning of 'new insight'.

Inclination to explore affects a student's likelihood of engaging with unfamiliar mathematics. This construct helps to explain why some students are inclined to challenge themselves to spontaneously explore novel mathematical ideas and other students are inclined to stay within the confines of the mathematical ideas presented and explained by the teacher. Inclination to explore has been linked to 'resilience' (Williams, 2003).

Resilience is '… the mechanisms and processes that lead some individuals to thrive despite adverse life circumstances.' (Galambos & Leadbeater, 2000, p. 291); or an "optimistic orientation" to the world characterized by a positive explanatory style where successes are perceived as permanent, pervasive, and personal, and failures as temporary, specific, and external (Seligman, 1995). Resilience is associated with exploratory mathematical activity because such activity can include many failures before a success. Optimistic (resilient) children perceived good fortune to result from their own endeavours rather than occur as a matter of chance, they see failures as temporary and as able to be overcome by their own endeavours. They also generalised successes as personal attributes and constrained failures to the specific situations in which the failure occurred (Seligman, 1995). These constructs are illustrated through the case presented.

'Autonomy' and spontaneity can be described in terms of the six social elements of the process of abstracting (Dreyfus, Hershkowitz, & Schwarz, 2001) by subcategorising these social elements (Williams, 2004). The six social elements of the process of abstracting are: 'control', 'elaboration', 'explanation', 'query', 'agreement', and 'attention'. Autonomous cognitive activity occurs when students control their focus of exploration and the pathways they take to explore new ideas (internal control, Williams, 2004). Spontaneous cognitive activity occurs during autonomous cognitive activity when the students identify their own inquiry (internal rather than externally directed attention), ask their own questions to clarify their understanding (internal query), answer their own questions (internal explanation), extend their own ideas (internal elaboration), and justify their own findings rather than require external agreement (Williams, 2004). Students can also ask questions to structure future exploratory activity (Cifarelli, 1999) (internal query, Williams, 2004). Schoenfeld, Smith, and Arcavi (1993, p 69)) describe an interaction between a student (IN), interviewer and a computer program (Black Blobs, similar to the program Green Globs used in this case) which can be used to illustrate an absence of student autonomy. The interviewer *directed* IN: a) to replace a number *she* wanted to trial with another number; and b) that she needed to use $y = 4x + 1$ when she input $2 = 4x + 1$ [external control, external elaboration, Williams, 2004]. The research question upon which this paper focuses is: What are the personal and social influences on creative student thinking and can they be encapsulated through activity during the process of developing mathematical insight?

Research Design

Data was generated as part of the international Learners' Perspective Study (see Williams, 2005). The thinking of students in six Year 8 classes (from Australia and the USA) was examined to find evidence of creative student thinking. Data was collected from each classroom over at least ten consecutive lessons. Three cameras simultaneously captured the activity of the teacher, a different pair of focus students each lesson, and the whole class. A mixed video image was produced during the lesson (focus students at centre screen and teacher as an insert in the corner). This mixed video image was used to stimulate student reconstruction of their thinking during individual post-lesson interviews. Students were asked to identify parts of the lesson that were important to them, and discuss what was happening, what they were thinking and what they were feeling. Through this process, students who explored mathematical complexities to generate novel mathematical ideas and concepts were identified and social and personal influences upon their thinking were made explicit through their discussion of the lesson video. Interviews, in conjunction with the lesson video, were used to identify intervals of time from when students first encountered difficulties with the teacher's task (or were curious about a complexity they identified) to when they spontaneously explored a complexity and developed an understanding of that complexity. Simultaneous analysis of student enacted optimism (Seligman, 1995) and the social and cognitive elements of the process of abstracting (Dreyfus, Hershkowitz, & Schwarz, 2001) assisted in identifying social and personal influences on student thinking.

The Context

Study of social and personal influences on students' mathematical thinking (Williams, 2005) was undertaken in four Australian and two US classrooms. The teachers selected were 'perceived to display good teaching practice by their school community'. The schools were geographically, socio-economically, culturally, and pedagogically diverse. Eden's constructing of new knowledge was selected to illustrate the findings. For detail of other cases, see for example Williams (2004) and Williams (2005).

Site and Subjects

Eden and Darius were Year 8 students in an inner-suburban Melbourne school with a diverse cultural mix and a 'good' academic reputation. They reported assisting each other in mathematics classes. In the lesson under study (Lesson 6), student pairs were seated side by side around the perimeter of the room, each with their own computer. Eden did not have a pair. Darius was the focus student for Lesson 6 and Eden was visible on the student camera at times when he moved across to view Darius' screen. On the Whole Class Camera, Eden was visible in the background but his computer screen was not visible. Although Eden's process of development of ideas and concepts was not captured on camera during Lesson 6, his elaboration of the mathematical structure he developed, his references to what stimulated his thinking, and the video record of his interactions with Darius were sufficient to capture the richness and originality of his thought processes. Eden's post lesson interview occurred after Lesson 8 (the next lesson on the same topic).

Lesson 6

Lesson 6 was an introduction to linear functions. The teacher stated the general form of a linear equation **y=mx+c** without explaining the role of the constants. Students were asked to work in pairs with the Green Globs game. Green Globs displays 13 globs (large dots) on a Cartesian plane on the computer screen. These globs are 'shot' using the trajectories of linear graphs. Higher scores are obtained where a function hits more globs (one for the first glob hit, two for the second, four for the fourth and so on).

Eden's Personal History

The class had not studied linear graphs previously. Eden had previously encountered linear graphs: "last year we did a bit on this stuff except I had forgotten *most* of it" [Line 87][1]. Eden's interview statement about having forgotten most of what he had learned was confirmed by his activity in Lesson 6. He was unfamiliar with 'gradient', did not recall the

[1] [Line 87], Line 87 of Eden's interview after Lesson 8. This notation is used for interview quotes.

general form of a linear equation, did not recognize equations that produced vertical and horizontal lines, and was unfamiliar with the term ‘intercept’.

Eden preferred problem solving to skills work.

> Problem solving's pretty good to work out and stuff … you've gotta (pause) use your mind a little bit more than just (pause) know how to add up sums and stuff [Lines 68-70]

Eden displayed frequent indicators of optimism (resilience) and no indicators of lack of optimism; he was inclined to explore. He perceived that prior learning could help in “a similar circumstance” (Success as Permanent). His perception that “work[ing] everything out for yourself … you will be able to think clearer” [Line 326] demonstrated he perceived failure as temporary and that individual effort would lead to success. His teacher did not recommend him for the accelerated group the following year [Teacher Interview] because she perceived him as an average student. The teacher’s perception did not fit with Eden’s high performance in problem solving on the Australian Mathematics Competition. Rather than perceiving failure (teacher’s perception of him as average) as a personal attribute (Failure as Pervasive), Eden constrained his use of the term average to the classroom in which that assessment was made. “*In this class* [researcher ‘s emphasis]” he was average and there was “no way to explain it” [Line 416] (Failure as Specific). Rather than perceiving this ‘failure’ as an attribute of himself, he identified the external factors associated with this failure (Failure as External, it was the teacher’s perception).

Results and Analysis

To report these results, a narrative of Eden’s activity in Lesson 6 has been constructed. Evidence from Eden’s interview is used to support this narrative.

Narrative of Eden’s Activity in Lesson 6

Students chose their directions of exploration, and discussed ideas with other students seated around them as they worked with Green Globs. Eden knew Darius was using trial and error (e.g., Darius to Teacher: “But how do you actually get them where you want them because I am just tapping in anything?” [14:37][2]). The teacher’s response to Darius was typical of her interactions with students in Lesson 6: “Well you will have to think about that” [14:46]. Eden’s main focus was on making sense of “angled” lines” (sloping lines) [19:10]. He frequently asked questions about *why* things happened rather than just focusing on how: “I didn't exactly know *why* it always happened like that” [Line 353].

Eden consulted with Darius on three occasions in Lesson 6 [12:50-17:20, 18:06, 24:23-27:23]. Eden then began to focus intently on a mathematical complexity that progressed his thinking [28:15]. Initially, Eden asked Darius: “What's the rule for that [sloping line on Darius’ screen]? That's the sort of angle” [12: 50]. Eden generated a horizontal line soon after (y=2+2) (probably missing the ‘**x**’ in the equation **y** = 2x + 2 suggested by Darius [17:20]). Eden exclaimed “Oh I get it- if you do two plus two is four” [18:06] as he watched the horizontal line appear on the screen. He had recognized that real number operations applied within equations. When Eden asked Darius for assistance with vertical lines [24:23], the pair interacted out of hearing of the microphone for three minutes. Each student then returned to his own computer and Darius generated a family of sloping lines

[2] [14:37], 14 minutes and 37 seconds into the lesson. This notation is used for times in Lesson 6.

by systematically manipulating numbers. In his interview, Darius reported that he used trial and error and did not know why the equations produced certain lines. Eden looked at these lines: “[to Darius] I don’t know how you get that” [27:56]. Darius was so involved in his score that he did not respond to Eden. Eden remained motionless as he watched the dynamic display Darius’ generated [27:58 - 28:15]. Eden then returned to his own computer and focused intently for seven minutes [Whole Class Camera].

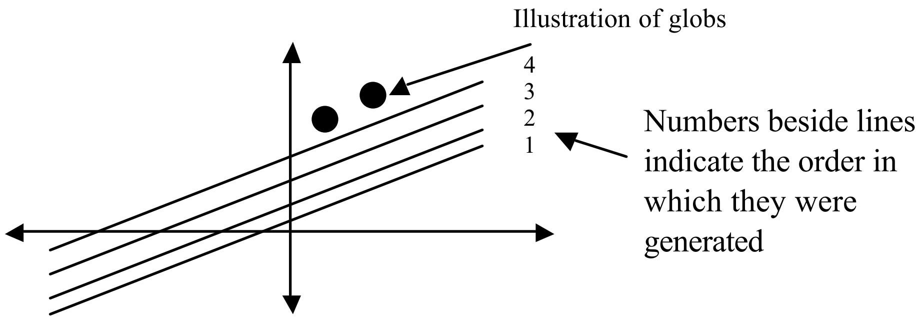

Figure 1 Darius’ computer screen generated progressively from 27:58 to 28:15

What did Eden attend to on Darius’ screen? Eden reconstructed his thinking:

> Then it's minus one to minus two- zero to minus one and then it keeps going *like that* (pause) so it is always *one ahead* [Line 139].

As interviewer, I did not understand this statement so asked for further clarification. The transcript below (in conjunction with Figure 2) elaborates Eden’s meaning.

> 246[3]. Eden …[4] well you get a graph like this [began to sketch graph] (pause) and basically you've got (pause) a little table like **x** and **y** [made table of values][5] (pause) and would be … minus two (pause) minus one (pause) or zero (pause) one (pause) and two (pause) [put x values in table] **y** is minus- starts off on minus three (pause) and you have got to (pause) … work out (pause) what **y** was (pause) which was which was minus three (pause) minus two (pause) minus one (pause) and zero (pause) and then (pause) *one* (pause) [put y values in table] and then the rule (pause) is ah (pause) would be (pause) um (pause) **y** (pause) equals (pause) **x** (pause) minus one.
>
> 249. Int How did you *find it*[6] though? (pause) You can show why- I can see that you are showing that it is the right one (pause) on what you say now but how did you *find* it?
>
> 250. Eden Well (pause) the graph's drawn up already (pause) for you to look at- that's the only help you get to answer
>
> 256. Eden … then at minus one you see it is minus two (pause) [focuses on co-ordinates of one point at a time] at zero minus one (pause) and so forth (pause)

Eden drew a graph and converted this graph to a table [Line 246], pointed to table cells as he explained the relationship he had found, and then summarised what he had demonstrated: “The rule … would be … y equals x minus one” [Line 246]. He then pointed to successive points on the graph to demonstrate the same relationship [Lines 250, 256].

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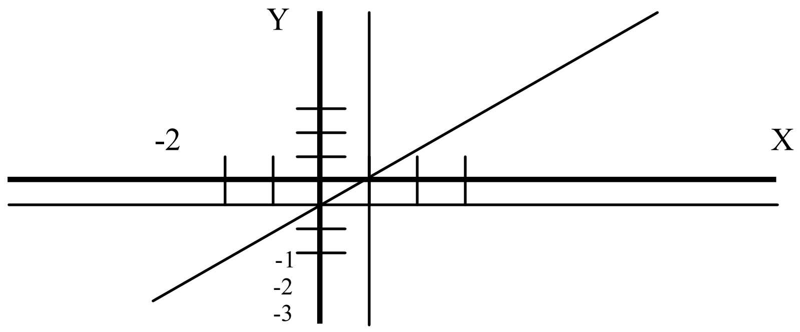
[3] Line numbers

[4] Part of the interview that was not relevant has been omitted.

[5] Researcher’s addition of information to clarify

[6] Italics for emphasis in interview



x	-2	-1	0	1	2
y	-3	-2	-1	0	1

Figure 2. Relevant features of Eden's sketch in his post-lesson interview

It appeared that the Green Globs program enabled Eden to evaluate his ideas by seeing whether his equations hit the desired globs. Eden's progressive (implicit or explicit) questions that structured the next part of his exploration seemed to be: Can this pattern help? Can I express this pattern in words? Can I express this pattern in symbols? Does it always work? At 35:12, Eden made a partially inaudible statement: "**y** is … cross … x". He used these same words in his interview as he described how the horizontal and 'angled' lines related to their respective equations: "y … crosses over with x". It is unclear which aspect of the relationship between horizontal and sloping lines Eden focused on. It could have been the y value relying on the value of x, or the presence of an x in the equation moving the y value away from a horizontal line, or something else. Whatever the focus, Eden was aware of these ideas after his interval of spontaneous exploration in Lesson 6.

Eden's Novel Mathematical Structure

In his interview, Eden demonstrated flexibility in moving between representations (table, graph, specific numerical co-ordinates, verbal, algebraic) as he clarified his meaning. He demonstrated mathematical activity valued by Schoenfeld, Smith and Arcavi (1993):

> Learning even simple knowledge in a complex domain means making connections, that is, a piece of knowledge is robust and stable to the extent that it is connected to other pieces of knowledge. (Schoenfeld, Smith, & Arcavi, 1993, p. 99)

Through his spontaneous exploratory activity (spontaneous pursuit of his exploration) Eden had abstracted new insight. Such insight was not achieved by the student IN who was of a higher year level and performed well on mathematics tests (see Schoenfeld, Smith and Arcavi, 1993). Eden had subsumed the graphical, numerical, and verbal relationships into an algebraic representation of the same relationship. He was able to unpack and explain his conceptual understanding when required. Subsuming was found to be key to the development of new insight in this study (e.g., see Williams, 2004, 2005).

Conclusions

Eden's case illustrates the six activities in the Space to Think that were found to be common to the students who creatively developed new mathematical knowledge in this study (Williams, 2005). Each of these students maneuvered Space to Think in a classroom in which the activity they undertook was not explicitly intended by the teacher. Had Eden worked in a pair as expected, he may not have had the cognitive autonomy to pursue his ideas. The six activities he exhibited (in the order presented) were:

1) Inclining to Explore

Eden's enacted optimism (see also Williams, 2006) by identifying and asking about aspects of the mathematical background he needed to begin his exploratory activity (e.g., the form of the equation) [Failure as Temporary, Failure as Specific]. He examined his attempts and found useful information (e.g., real number operations in equations) [Failure as Specific].

2) Spontaneously Identifying Complexity

By Looking-in on Darius' screen, Eden identified a complexity he had not been aware of previously (a relationship between the ordered pairs) [Failure as Specific].

3) Manoeuvred Cognitive Autonomy

By working alone rather than with Darius (who focused on trial and error), Eden was able to explore this pattern and construct new insight.

4) Autonomously Accessing Mathematics

Eden assembled cognitive artifacts he had possessed previously (e.g., understanding of Cartesian Co-ordinates) and cognitive artifacts he formulated earlier in the lesson (e.g., real number operations apply in equations). In addition, he Looked-in to develop new mathematical ideas (e.g., there is a relationship within the ordered pairs).

5) Spontaneous Pursuit

The subsuming process described earlier occurred as a result of Eden building-with the cognitive artifacts he assembled and finally constructing new knowledge. By using real number operations within equations, and building-with the pattern he had identified to express it in various representations, he recognized he had produced the algebraic form of the graph through the activity he had undertaken. Initially he moved from representation to representation as he formulated the same relationship within each (building-with). He finally developed insight into why the equation 'worked' (building-with nested within constructing). He subsumed the various representations into the algebraic representation and was able to 'unpack' this representation to show he understood its meaning.

6) Structuring Questions

Types of questions Eden asked were identified earlier. Other cases provide more detailed evidence of this activity (e.g., Williams, 2004).

Space to Think

The crucial nature of autonomy and spontaneity in creative student thinking is confirmed by the nature of activities that emerged through the Space to Think. They illuminate the sensitivity of creative activity to small social changes (*Spontaneously* Identifying Complexity, Cognitive *Autonomy*, *Autonomously* Accessing Mathematics, *Spontaneous* Pursuit). The conclusions drawn from the five cases are strengthened by the diversity of the situations studied. Further research is required to find whether these six activities are always present in the Space to Think, and whether other activities are also present. The crucial nature of resilience (inclination to explore) in creative activity is identified and the need for research into resilience building activity is highlighted. Seligman (1995) has shown that resilience is built by engineering flow situations (Csikszentmihalyi & Csikszentmihalyi, 1992). Research is required to identify how to structure curricula which provides resilience-building opportunities. The role of Looking-in as an activity that can compensate for absence of appropriate cognitive artifacts informs task design. Areas of research suggested through this study have the potential to optimise mathematics learning.

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