Describing and Exploring the Power of Relational Thinking

Max Stephens
The University of Melbourne
m.stephens@unimelb.edu.au

Blanton and Kaput (2001) encourage teachers, especially in the primary school, to grow "algebra eyes and ears" (p. 91). This is not an easy task when teachers' vision has for so long been restricted to thinking of arithmetic primarily as computation. This paper looks at how forms of thinking which do not rely on computation can be used with addition and subtraction number sentences. It looks at how these powerful forms of thinking are used by students in Years 5 to 7 in two schools, what distinguishes this thinking from computational thinking, and how consistently it is used by students across different problem types.

Relational Thinking

The focus of this paper is to examine some powerful forms of thinking that young children apply to addition and subtraction sentences. These forms of thinking are described in this paper as *relational* to distinguish them from *computational* forms of thinking that all children are taught to use. Essentially, *relational* thinking depends on children being able to see and use possibilities of variation between numbers in a number sentence. When children in primary school were presented with the following written scenario in Figure 1 one Year 6 student said, "Loretta just knows that they both add up to 63". A Year 5 student said: "Loretta can do this because she did it in her head". Neither student could explain why Loretta did not need to add the numbers in order to know that she was correct without using an explanation based on computation.

Loretta has written the following number sentence 34 + 29 = 33 + 30 She did not have to add up the numbers to know this. Why?

Figure 1: Loretta's Thinking

There are other students who can explain what they think Loretta has done without assuming that she performed a computation. Here are two examples from Year 5 students. One student drew an arrow connecting 29 on one side to 30 on the other and attached +1 to this arrow. This student also drew an arrow underneath connecting 34 to 33 attaching a _1 to this arrow. Starting with the 29, the student explained, "It increases by 1 to give 30, so 34 has to decrease by 1 to give 33". A second Year 5 student wrote the two numbers, 33 + 30, and drew an arrow connecting 30 and 33. No number was attached to the arrow, but the student said: "If one unit moves from the 30, the other number becomes 34".

By considering numbers either side of the equal sign, the first student is able to explain that if one number increases (or decreases) by 1, the other number has to decrease (or increase) by 1 in order for the two sides to be equal. The second student is able to see a variation of 1 from one number to the other when two numbers are added together. Each student is able to show that numbers in an addition sentence can be varied without altering the truth of the sentence. These two students do not rely on computational thinking to explain what Loretta has done.

This contrast between relational thinking, underpinned by a capacity to see possibilities of variation between numbers in a number sentence, and computational thinking where children have to perform a calculation, is shown clearly when children are asked to find a missing number in a number sentence. The questions shown in Figure 2 were given to 301 students in Years 5 to 7 in two Melbourne schools.

Write a number in each of boxes to make a true statement. Explain your working.
73 + 49 = 72 + ❑ 99 – ❑ = 90 – 59 746 – 262 + ❑ = 747

Figure 2: Sample Missing Number Sentences

In the first item, some students first calculated the sum of the numbers on the left-hand side to give 122, and then worked out what needed to be added to 72 to give the same sum. They found the missing number to be 50, either by adding on from 72 to get 122 or by subtracting 72 from 122. In the second item, some students first showed that the right-hand side was equal to 31 and then found what number subtracted from 99 would give 31, giving an answer of 68. In the third item, children first calculated 746 – 262 giving 484, and then found the number which added to 484 gives 747. In this way, they got a correct result of 263 for the missing number.

Contrast these forms of careful *computational* thinking, which many students successfully executed, with *relational* approaches which relied on seeing possibilities of variation between numbers on either side of the equal sign. As Marton and Tsui (2004) point out, "Mathematical content which appears superficially the same can have different characteristics for different students" (p. 35). For the first item, some students, noting that 72 and 73 differ by 1, decided that the missing number was 1 more than 49. In item 2, students argued that, since 99 is 9 more than 90, the missing number has to be 9 more than 59, because each side represents a difference. In item 3, some students argued that, since 747 is 1 more than 746, this increase of 1 is the result of subtracting 262 and adding another (the "missing") number which must 263.

Equivalence in Research Literature

Many researchers have focussed on the idea of equivalence to explain how relational thinking is different from computational thinking. Kieran (1981), for example, pointed to the importance of seeing the equal sign as an indication of equivalence or a balance, and noted that many children in primary school still think of the equal sign as a direction to find the answer. Carpenter and Franke (2001) also argued that understanding equivalence between terms and operations each side of the equal sign enables children to think relationally. These two authors as well as Blanton and Kaput (2001) use the expression "algebraic thinking" to describe what is here called relational thinking.

Carpenter and Levi (1999) also point out that understanding equivalence is a necessary condition for solving number sentences such as 13 + 8 = ❑ + 10. They say that children who think that the equal sign represents the *operation* of finding an answer may well think that the missing number in the preceding sentence is 21, appearing to disregard the presence of the 10. Successful computational thinkers do rely on a correct understanding of equivalence. They reason that the since two numbers on the left add to 21, the two numbers on the right must also add to 21. Since one number is 10, the missing number has to be 11. This sense of equivalence being used here may be called *equivalence of results*.

Relational thinkers seem to use equivalence in another way. What is interesting is that they are able to keep the two numbers either side of the equal sign as *uncalculated* pairs. Collis (1975) refers to this as *acceptance of lack of closure*. Not having to rely on equivalence of results, relational thinkers can work with uncalculated pairs in equivalent expressions. Then it is possible to ask, in the above example, what will happen to the first number (shown by o on the right) if the 8 increases to 10. It is not correct to say that relational thinkers first *abstain* from 'closing' some operations and then look at the size and direction of the variation to find a missing number. It would seem more correct to say that relational thinkers are able to abstain from closing some operations *because* they see possibilities of variation between the numbers in an equivalence relation where the direction and size of the variation depend on the numbers and operations involved.

Possibilities of variation are more easily discerned when the numbers are close to each other like 99 and 90, or 72 and 73, or 746 and 747. If one had a number sentence like 109 + 76 = 23 + ❑, it may be just as quick to add the left hand side to give 185 and then ask what has to be added to 23 to give 185, resulting in answer of 162. It is true that, since 23 is 53 less than 76, the missing number must be 53 more than 109. (One could also say that, since 109 is 86 more than 23, the missing number is 86 more than 76.) While mathematically true, these variations are not easily discerned by most children (or adults for that matter); or, if they are, they are not seen as offering a more expeditious route to getting an answer.

The notion of equivalence is clearly important. However, it will be argued that understanding equivalence *permits* but does not define relational thinking. What seems to mark relational thinking is the capacity to use possibilities of variation in number sentences. Watson and Mason (2004) point to the importance of enabling students to discern dimensions of variation in which some elements in mathematical sentences change while other elements remain unchanged. To use these possibilities in the above three missing number sentences, students must know the *direction* in which variation occurs. Discerning possibilities of variation in specified *directions* is the key to relational thinking.

Directions of Variation

More is required to discern possibilities of variation in item 1 in Figure 2 than for students to notice that 73 is one more than 72. Students need to know what this difference means for finding the value of the missing number. They need to know the *direction* in which the missing number will change. In this example, it has to be 1 more than 49 for both sides to be equivalent. In the second item, it is not enough to see that 99 is 9 more than 90. One has to be able to discern that the missing number has also to be 9 more than 59 for the difference to remain the same. Just seeing a difference of 9 between the first numbers on either side of the equal sign is of no use unless one also knows the direction in which 59 needs to vary in order to keep the equivalence.

The directions or possibilities of variation in 73 + 49 = 72 + ❑ are quite different from 90 – 59 = 99 – ❑. In the first item, some children may say that one has been removed from 73 and so has to be added to 49 to keep the sum the same. The reasoning in the second item is more complex. For example, some children may say: "For the difference between the two numbers to remain the same, an increase of 9 in the first number has to be matched by an increase of 9 in the second number". Other children may say: "If the first number increases to 99, then leaving the second number as 59 changes the answer. So to keep the answer the same the second number has to increase by 9 as well, giving 68."

The direction of variation in the subtraction item is not the same as for the addition item. If 90 increases to 99, then reducing the second number by nine to 50 gives an incorrect result: 90 – 59 is not equivalent to 99 – 50. Relational thinking appears to be characterised by a capacity to see possibilities of variation between the numbers where the possibilities embody specific *directions* of change depending on the operations involved. Knowing that one number is greater or less than another number is no use unless one also knows the direction of change implied by these differences.

Gathering Data from the Two Schools

Design of Tasks and Task Demands

Three groups of tasks shown in Figure 3 were given to 301 students in two schools using a pencil-and-paper questionnaire administered in regular class time. In introducing the questionnaire, classroom teachers reminded the students that:

> "This is not a test. It is a questionnaire prepared by researchers … looking at how students read interpret and understand number sentences. For most of the questions there is more than one way of giving a correct answer. Please write your thinking as clearly as you can. Write your thinking in the space provided after each question and don't feel that you have to write a lot."

Students had sufficient time to compete all three groups of questions. In both schools, the questionnaire was given to two classes at each of Years 5 and 6. In Year 7, it was given to two classes in School A and to three classes in School B. Each group of problems was introduced with the words: "Write a number in each of boxes to make a true statement. Explain your working".

Group A (one page)	Group B (one page)	Group C (two pages)
23 + 15 = 26 + ❑	39 – 15 = 41 – ❑	746 – 262 + ❑ = 747
73 + 49 = 72 + ❑	99 – ❑ = 90 – 59	746 + ❑ – 262 = 747
43 + ❑ = 48 + 76	104 – 45 = ❑ – 46	
❑ + 17 = 15 + 24		

Figure 3: Three Groups of Missing Number Sentences

Group A and Group B problems were given on separate pages with sufficient space for students to show their working. The two problems in Group C were given on separate pages with the same introduction in each case.

For students who used computational procedures to find the value of the missing numbers, Group A was expected to be easiest, involving an initial addition and then a subtraction. Group B was expected to be more difficult, leading in some cases to arithmetical errors because students were required to perform an initial subtraction followed by a second subtraction, and Group C was expected to be the most difficult to solve correctly, involving calculations with three-digit numbers.

For students who used a relational approach, Group A was thought to be easiest. It was expected that some students might use a relational approach in Group B, but mistake the direction of variation. A relational approach to Group C questions was expected to be less prone to errors than approaches based on computation.

In the two schools where questionnaires were given the teaching of computational algorithms forms an important part of the curriculum. While relational approaches were generally not given emphasis in School A, it is not possible to say that children had not been exposed at some time to using possibilities of variation to solve number problems. For example, teachers do show children that adding 99 is equivalent to adding 100 and then subtracting 1 from the result. Although one teacher in School B did make explicit use of relational strategies, the other teachers said that they did not use these approaches regularly.

Scoring Procedures

Each group of problems was scored using a five-point scale shown in Figure 4. This allowed a single score to be assigned to each group of questions even when children may not have solved each question in the same way. The following scoring scheme which had been validated for an earlier study (Stephens, 2004) was applied to Groups A, B, and C.

0 – arithmetical thinking evident for all questions; for example, through evidence of progressive calculations and use of algorithms to obtain relevant totals for additions and subtractions, even where these approaches may have resulted in incorrect answers, and no evidence of any relational thinking; also where an answer only has been given without any working shown to indicate what method has been used 1– a clear attempt to use relational thinking in at least one question, but not successfully executed (e.g. in Group B by giving answers of 13, 15 and 103) 2 – relational thinking clearly shown in one question and successfully executed, even if the other problems are solved computationally or using incorrect relational thinking 3 – relational thinking clearly shown in at least two questions and successfully executed, but where the remaining question or questions are not solved relationally or solved using incorrect relational thinking 4 – all questions are solved clearly and successfully using relational thinking, computational solutions may also be provided.

Figure 4: Scoring Rubric

Relational thinking was evident when, for example, arrows or diagrams were used to compare the size of numbers either side of the equal sign; or verbal descriptions used to compare the relative size of numbers either side of the equal sign; and where these arrows, diagrams or descriptions were used in chain of argument, based on uncalculated pairs, to find the value of a missing number; even if computational solutions were also provided. For each group of questions a benchmark sample was prepared, illustrating each score.

Results of Scoring

Scoring was double-checked using an additional marker, and any discrepancies reconciled. The results are shown in Table 1.

Table 1:
Distribution of students' scores by year level and school (School A, School B)

Group A	Score 0	Score 1	Score 2	Score 3	Score 4	N =
Year 5	*71%* 45%	*0%* 2%	*10%* 4%	*5%* 15%	*15%* 34%	*41*, 50
Year 6	*60%* 34%	*0%* 2%	*11%* 4%	*4%* 12%	*25%* 48%	*45*, 50
Year 7	*61%* 21%	*0%* 2%	*14%* 4%	*5%* 16%	*20%* 57%	*44*, 71
Group B						
Year 5	*75%* 52%	*5%* 26%	*15%* 2%	*0%* 0%	*5%* 20%	*41*, 50
Year 6	*64%* 32%	*18%* 24%	*5%* 8%	*5%* 6%	*9%* 40%	*45*, 50
Year 7	*66%* 25%	*7%* 17%	*9%* 4%	*0%* 7%	*18%* 47%	*44*, 71
Group C						
Year 5	*80%* 70%	*3%* 6%	*10%* 8%	*0%* 0%	*7%* 16%	*41*, 50
Year 6	*78%* 48%	*0%* 5%	*4%* 17%	*2%* 4%	*16%* 26%	*45*, 50
Year 7	*75%* 27%	*0%* 2%	*7%* 18%	*2%* 7%	*16%* 46%	*44*, 71

Discussion and Analysis

In School B, the percentage of students attaining Score 4 is much higher at every Year level than in School A. (In School B there were some variations between class groups.) The percentage of students attaining Score 4 in School B also increases at every Year level. Focussing on these highest scores may not, however, be the most helpful way of looking at either school. If successful relational thinking is shown by a score of 2 or more, this measure shows growth in relational thinking between successive year levels in each school.

In School A, looking at the percentages of students scoring ≥ 2 in each Year level, gives 30%, 40% & 39% for Group A; 20%, 18% & 27% for Group B where the dip in Year 6 is explained by the 18% of students who scored 1 on these problems. For Group C, a score of ≥ 2 was shown by 17%, 22% & 25% across the three Year levels. For School B, the comparable percentages of students scoring ≥ 2 are 53%, 64% & 77% for Group A; 22%, 54% & 58% for Group B; and 24%, 47% & 71% for Group C. In School B, the percentages of students who used "failed relational" (Score 1) thinking for Group B questions was 26% for Year 5, 24% for Year 6, and 17% in Year 7. In both schools, Score 1 on Group B was invariably associated with a score of ≥ 2 on Group A and/or Group C.

Figure 5 gives examples of relational thinking showing unequivocal understanding of equivalence and correct directions of variation between uncalculated pairs. Two student responses are given for each group of items.

If I take 2 from 17 and add 2 to 22, it is the same as the number sentence after it. (School A, Year 6) In 43 + o = 48 + 76, 43 to 48 is +5, 81 to 76 is –5. These are equivalent, as you've done the same action to both sides. (School A, Year 7)

As 99 is 9 more than 90, the missing number must be 9 more than 59. Therefore the answer is 68. (School B, Year 5)
I added 1 to 104 and 45. As long as I add the same number to both, it (104 – 45) will stay equivalent. (School B, Year 6)

746 is one less than 747, so 262 is one less than the answer. My answer is 263. (School A, Year 5)
746 is I unit less than 747, so if you add 263 you will only need to minus 1 unit less than 263 for the equation to be equal on both sides. (School B, Year 7)

Figure 5: Selected students' responses showing relational thinking

There is no need to give examples of computational thinking. It is important to note that computational thinking, for the reasons stated earlier, was often prone to errors. For example, in Year 5 at School A, among the thirty-three students who tackled Group C using computation, only seven students calculated correct answers to both parts of Group C. Nine students failed to get a correct answer to either part. Seventeen students obtained one correct answer, usually to the first item, 746 – 262 + ❑ = 747. The second item 746 + ❑ – 262 = 747 proved more difficult for computational thinkers at all Year levels.

Stability of Thinking

How consistent were students in their use of relational or computational approaches across the three groups of problems? Students were classified into three groups: those students who used relational strategies across all three groups of problems (SR–Stable Relational); those students who used only arithmetical or computational approaches across all three groups of problems (SA–Stable Arithmetical); and those students whose thinking was not consistent across the three groups (NS–Not Stable). The following rule was used:

SR: if a student scored ≥ 1 on each of Group A, B, and C

SA: if a student scored 0 on each of Group A, B, and C

NS: if a student scored ≥ 1 on one or two of Group A, B, or C; and 0 on the other(s).

A criterion of ≥ 1, instead of ≥ 2, across the three groups as evidence of stable relational thinking was justified because a score of 1 on Group B was without exception associated with successful relational thinking (≥ 2) for Group A and/or Group C questions. Table 1 shows that, aside from responses to Group B, a score of 1 was infrequent.

Results of Stability Analysis

These results are shown separately in Figure 6 for each school. Any attempt to deal with the whole cohort would disguise important differences across the two schools.

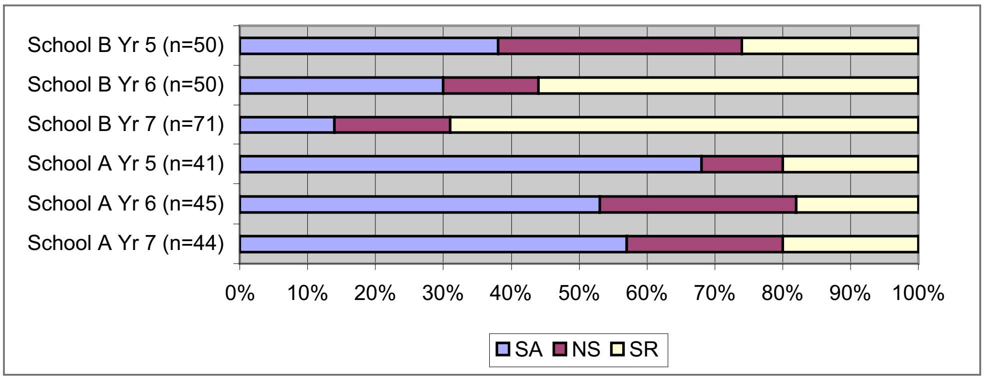

Figure 6: Results of Stability Analysis

In School A, the proportion of SR appears to change very little over the three years. However, the biggest difference between Year 5 and Year 6 is the proportion of SA being 15% lower in Year 6, and a corresponding rise in the NS classification. This suggests an emergence of relational thinking between Year 5 and Year 6.

In School B, the proportion of SR in Year 5 is only slightly higher than that of School A, but already in Year 5 there is a higher proportion of students (NS) who in Year 5 show evidence of relational thinking than in School A. Comparing Year 6 to Year 5, the proportion of SR is more than double, with a smaller proportion of NS in Year 6 than in Year 5, and only a small drop in the proportion of SA. The proportion of SA is smaller in

Year 7 with a greater proportion classified as SR and NS. This may well be explained by explicit teaching of relational approaches in School B by one specialist mathematics teacher who takes a class at all three levels.

Conclusion

Several authors already cited argue that the use of relational thinking to solve number sentences of the kind shown above can be called algebraic. Pimm (1995) would agree, arguing that "Algebra, right back to its origins, seems to be fundamentally dynamic, operating on transforming form. It is also about equivalence, something is preserved despite apparent change" (p. 88). Fujii and Stephens (2001) also suggest that when children show that the numbers in sentences like 73 + 49 = 72 + 50 can vary without affecting the truth of the expression, they are engaging in generalised numerical thinking which may be called *quasi-variable* thinking. Of course, primary school children have not yet learned that 73 + 49 is always equivalent to $(73 - a) + (49 + a)$. However, when children create other equivalent arithmetical expressions, they are using *some* possibilities of variation that can be applied to 73 + 49. Children who use relational thinking to generate equivalent expressions to 73 + 49 may not understand the full *range of variation* embedded in the formal statement $73 + 49 = (73 - a) + (49 + a)$ where a can represent whole numbers, and rational and negative numbers as well; and where $(73 - a) + (49 + a)$ can also be written as $(73 + a) + (49 - a)$. But, their successful generation of specific instances using relational thinking can be described as *quasi-variable* thinking because its *range of variation* is limited. This appears to be a pre-cursor of formal algebraic thinking. Its capacity to generate equivalent numerical expressions makes it different from computation.

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