Identifying Key Transition Activities for Enhanced Engagement in Mathematical Modelling

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Three current interpretations of the term 'mathematical modelling' as it is used in mathematics education are described. The modelling cycle appropriate to one of these interpretations forms the basis for research into blockages that emerge in the solution process for problems with real world connections. The development of a framework documenting key elements that enable (or disable) progress during transitions between phases in the modelling process is described, and a selection of elements illustrated. Associated implications for learning and teaching are discussed.

The term *mathematical modelling* as it is used in curricular discussions and implementations does not have a single meaning. One major interpretation uses mathematical modelling primarily for the purpose of motivating, developing, and illustrating the relevance of particular mathematical content. "We recognise that extensive student engagement in classroom modelling activities is essential in mathematics instruction only if modelling provides our students with significant opportunities to develop deeper and stronger understanding of curricular mathematics" (Zbiek & Conner, in press). *Emergent modelling,* as a conceptual framework and modus operandi (Gravemeijer, 1999), is located essentially within this purpose. The perspectives of Lesh and Doerr (2003) and English (2003) encompass this view but extend beyond to include elements of the second perspective that follows. This second perspective, which we favour, does not view applications and modelling primarily as a means for achieving some other mathematical learning end, although at times this is a valuable additional benefit. Rather this view is motivated by the desire to develop skills appropriate to obtaining a mathematically productive outcome for a problem with genuine real-world connections (e.g., Blum, 2002; Galbraith, Stillman, Brown, & Edwards, in press; Pollak, 1997). Here the solution to a problem must take seriously the context outside the mathematics classroom, within which the problem is located, in evaluating its appropriateness and value. It is a view that has characterised the International Conferences on the Teaching of Mathematical Modelling and Applications and the curricular call to arms on the part of those such as Pollak (in press). It was also a central emphasis in the Discussion Document for ICMI Study 14 (Blum, 2002). (We do not attempt here to encompass various other idiosyncratic interpretations of the term mathematical modelling as used in some localised curricular implementations.) While the above approaches differ in the emphases they afford modelling in terms of its contribution to student learning, they generally agree that modelling involves some total process that encompasses formulation, solution, interpretation, and evaluation as essential components.

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Within these approaches links between the real and mathematical worlds are maintained, even though they may at times be somewhat strained. It has become clear however that the term mathematical modelling is increasingly being used in a much more restricted sense, to mean nothing more than fitting curves to sets of data points, and the increasing use of technology means that this issue may become increasingly pervasive. The following example illustrates associated implications for the integrity of models. Data showing minutes of sunlight for two Australian cities are provided at intervals of 4-weeks for a calendar year. The purpose of the problem is to obtain mathematical functions that describe the data and to use them to make various comparisons about aspects of life in the respective cities. Noting that the data, when plotted, suggest a translated and dilated cosine function of the form $y = a + b\cos(\frac{2\pi}{T}(x+c))$, the period may be reasonably taken as 365 (days), and the minimum and maximum values estimated from the data by noting that these occur respectively on June 21, and December 21. A resulting model for one of the cities, that fits the data well, has the equation $y = 730+158\cos(\frac{2\pi}{365}(x+11))$. Now the TI-83 Plus graphing calculator has a trigonometric curve fitting facility among its regression options that generates an immediate function of best fit that is technically a closer match than the above. Expressed in the same form it has the following equation: $y = 731+155\cos(\frac{2\pi}{377}(x+17))$. Interpretation of this 'closer fit' then infers a 'year' of 377 days, with the longest day around December 15, outcomes that fail the fundamental test of real-world validity. A 'model' generated by this means is a purely technical artefact whose parameters vary with the particular data set, and which can be generated in complete ignorance of the principles underlying the real situation. At another level it raises a profound theoretical issue — the relative authority of empirical data versus theoretical structure. While curve fitting is an important component skill, using curve fitting as a synonym for mathematical modelling is an aberration of modelling. In particular, the subversion of the requirement of testing against reality by making choices based on the menus of graphing calculators or computers represents a substantial distortion of the purpose of modelling, and leads to both inappropriate modelling habits and outcomes.

Modelling Process

Various diagrammatic representations of the modelling process, as it applies within the second perspective, are common in the literature (e.g., Merrill, 2003) and most of these are relatives or descendants of a diagram initially provided by the Open University (UK). Such diagrams illustrate key stages in an iterative process that commences with a real world problem and ends with the report of a successful solution, or a decision to revisit the model to achieve a better outcome. The purpose is to provide a scaffolding infrastructure to help modellers through stages of what can appear as a challenging and opaque task. It is based on procedures that real world problem solvers undertake. However, as pointed out by Blum and Leiß (in press), when interests in teaching and learning are also central we need a version more oriented towards the *problem solving individual,* to give not only a better understanding of what students do when solving (or failing to solve) modelling problems, but also a better basis for teachers' diagnoses and interventions. Figure 1 contains a structure that encompasses both the task orientation of the original approach, and the need to capture what is going on in the minds of individuals as they work, often idiosyncratically, on modelling problems.

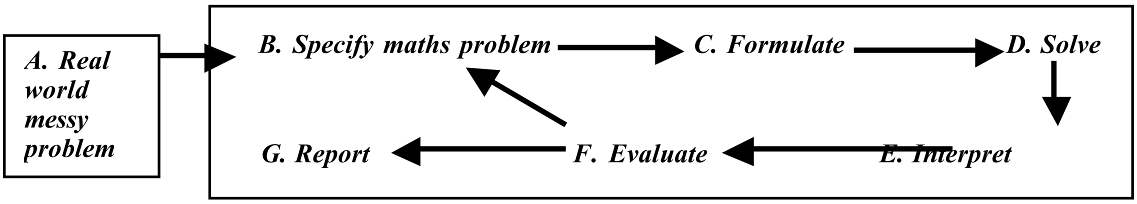


Figure 1. Modelling process chart.

The respective entries A to G represent stages in the modelling process, and the arrows signify transitions between the stages. The total problem solving process is described by following these arrows clockwise around the diagram from the top left. It culminates either in the report of a successful modelling outcome, or a further cycle of modelling if evaluation indicates that the solution is unsatisfactory in some way.

Now, adding an educational focus, we turn attention to the kinds of mental activity that individuals engage in as they move around the modelling cycle. As the term ‘activity’ suggests, these can be expressed in terms of *verbs* that describe what happens as modellers achieve a successful transition (or not) from one modelling stage to the next, (where there is special interest in identifying blockages that impede progress). At a theoretical level these may be thought of as generic activities as illustrated for the transitions below.

A → B: Understanding, structuring, simplifying, interpreting context
B → C: Assuming, formulating, mathematising
C → D: Working mathematically
D → E: Interpreting mathematical output
E → F: Comparing, critiquing, validating
F → G: Communicating, justifying, report writing (if model is deemed satisfactory) **OR**
F → B: Revisiting the modelling process (if model is deemed unsatisfactory).

Research Focus

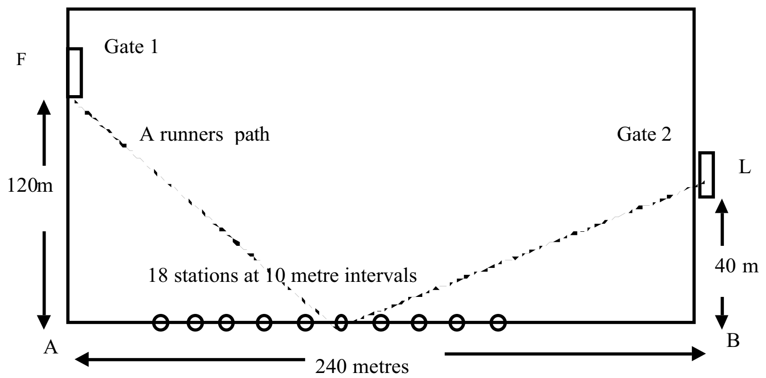
Our focus is located at the level of the actions of individuals while learning and applying modelling skills in a Technology-Rich Teaching and Learning Environment (TRTLE) (Brown, 2005). A classroom in which mathematical modelling is being enacted is a varied and unpredictable place, featuring intense activity, problematic times when blockages occur, and spontaneous and unforeseen actions by students (and teachers) engaging with new material and challenges. Such a culture is central to the process of teaching mathematical modelling skills, where successive implementations even by the same teacher can vary substantially in detail. The nature of the world outside the classroom such that real problem data are usually messy. Consequently, the appropriate use of technology is central to our purpose, and its integration with mathematics within the modelling process is creating essential challenges about which more needs to be known. In particular we focus on critical points occurring within transitions between stages in the solution process described in the previous section.

There are two main research goals:

1. 1. To identify and classify critical aspects of modelling activity within transitions between stages in the modelling process.
2. 2. To identify pedagogical insights, for implementation through such activities as task design and organisation of learning.

Data for this paper were generated within RITEMATHS, an Australian Research Council funded project of the University of Melbourne and the University of Ballarat with six schools and Texas Instruments as industry partners. The research being undertaken is part of a design experiment (English, 2003) in its second cycle at the time of this data collection and involved implementation of two tasks in one school. This school has been developing a lower secondary mathematics curriculum providing opportunities for engagement in extended investigation tasks set in real-world contexts. A focus to date has been in Year 9 when students (14–15 year olds) are required for the first time to have laptop computers and graphing calculators. Intensive data were generated, in the form of student scripts (24 and 28 respectively), videotaping of teacher and selected students, video and audio-taped records of small group collaborative activity, and selected post-task interviews (8 and 4) respectively. In order to identify and document characteristic levels of performance; occurrence or removal of blockages; use of numerical, graphical, and algebraic approaches; quality of argumentation; and the respective interactions between modelling, mathematical content, and technology, these data were entered into a NUD.IST database (QSR, 1997) and analysed through intensive scrutiny of the data to develop and refine categories related to these themes. Illustrations used in this paper are drawn from the analysis of the implementation of one of the tasks, *Cunning Running* (Figure 2), that occupied approximately one week of class time.

Cunning Running: In the annual "KING OF THE COLLEGE" Orienteering event, competitors choose a course that will allow them to *run the shortest possible distance,* while *visiting a prescribed number of check point stations.* In one stage of the race, the runners enter the top gate of a field, and leave by the bottom gate. During the race across the field, *they must go to one of the stations* on the bottom fence. Runners claim a station by reaching there first. They remove the ribbon on the station to say it has been used, and other runners need to go elsewhere. There are 18 stations along the fence line at 10 metre intervals, and the station closest to Corner A (station 1) is 50 metres from Corner A. The distances of the gates from the fence with the stations are marked on the map.



TASK: For the station on the base line closest to Corner A, calculate the total path length for the runner going Gate 1 – Station 1 – Gate 2. Use Lists in your calculator to find the total distance across the field as 18 runners in the event go to one of the stations, and draw a graph that shows how the total distance run changes as you travel to the different stations. Observe the graph, then answer these questions. Where is the station that has the shortest run total distance? Could a 19th station be entered into the base line to achieve a smaller total run distance? Where would the position of the 19th station be? If you were the sixth runner to reach Gate 1, to which station would you probably need to travel? What is the algebraic equation that represents the graph pattern? Draw the graph of this equation on your plot of the points. If you could put in a 19th station where would you put it, and why?

Figure 2. Cunning Running Task.

The key steps in the solution of *Cunning Running* follow. The solution involves the calculation of the total path as the sum of two segments, followed by graphing, construction of an algebraic model, verification, interpretation, and the search for a nineteenth station optimally located. Total path, for example, is given by $L = \sqrt{(14400+x^2)} + \sqrt{(1600+(240-x)^2)}$ where x is the distance to a station from corner A. Figure 3 shows a typical spreadsheet graph produced by students to show the different values of path length calculated for the separate checkpoint stations obtained, for example, using the LIST facility on a graphing calculator. The equation can be checked, using the function graphing and the plotting facilities of a graphing calculator to show the graph for L in terms of x passes through the scatter plot of the points. Deciding which checkpoint station to use (if the sixth runner), and selecting a site for the 19th station, are inferred from the behaviour of L in terms of x, as displayed in the graphical output.

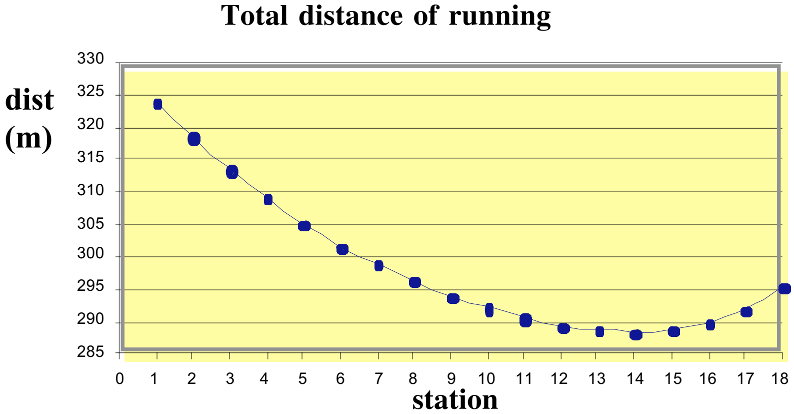


Figure 3. Spreadsheet chart for length of path.

Some Outcomes

The structural framework (Figure 4) consists initially of the transitions (from Figure 1) of which four have been included for present purposes. Initially the contents of the respective sections are empty — the production of the contents is described below. Each second level entry has two parts. The left-hand statement is a generic descriptor for a particular category of modelling activity that presents a blockage if absent or unsuccessful. The right-hand statement in capitals illustrates this using an example from *Cunning Running*. Below we elaborate more using samples from observation and student work.

Messy real world situation → real world problem statement

To assist in the early specification and formulation stages (avoiding an early blockage), the teacher provided a supporting dynamic geometry animation of the task which students watched (1.1).

Interviewer: The Friday before you did the task you saw a GSP animation of the task.

Gary: [showing the movement with his fingers using the task diagram] Awh, and he moved the bar. It just, it showed um, um, I understood it. It was just showing you the length and how you actually got it, the area … . It showed everything that you needed really. … it was just good to see it in front of you and it doing its own little business [indicating the movement of the station along the base line with his fingers again].

Identification of strategic entities to form a basis of model building is the first analytic task

(1.3), followed by specification of the correct element of this strategic entity. Here the strategic entity is length, and the key element a compound distance to be constructed from other components of the situation. Prior to this identification, simplification (representation of paths by straight lines) provides a basic structure for the problem context (1.2).

1. MESSY REAL WORLD SITUATION → REAL WORLD PROBLEM STATEMENT
1.1 Clarifying context of problem [ACTING OUT, SIMULATING, DISCUSSING PROBLEM SITUATION]
1.2 Making simplifying assumptions [RUNNERS WILL MOVE IN STRAIGHT LINES]
1.3 Identifying strategic entit(ies) [RECOGNISING LENGTH OF LINE SEGMENT AS THE KEY ENTITY]
1.4 Specifying correct elements of strategic entit(ies) [IDENTIFYING SUM OF TWO CORRECT LINE SEGMENTS]
2. REAL WORLD PROBLEM STATEMENT → MATHEMATICAL MODEL
2.1 Identifying dependent and independent variables [TOTAL RUN LENGTH AND DISTANCE FROM CORNER]
2.2 Representing formulae in terms of 'knowns' [LENGTH EXPRESSED IN TERMS OF FIELD EDGE DISTANCES]
2.3 Realising independent variable must be uniquely defined [X-CANNOT BE DISTANCE FROM BOTH A AND B]
2.4 Choosing technology to enable calculation [RECOGNISING HAND METHODS ALONE ARE IMPRACTICAL]
2.5 Choosing technology to automate formulae for multiple cases [LISTS HANDLE MULTIPLE X-VALUES]
2.6 Choosing technology to produce graphical output [SPREADSHEET OR GRAPHING CALCULATOR]
3. MATHEMATICAL MODEL → MATHEMATICAL SOLUTION
3.1 Generating appropriate formulae [$L = \surd(14400+x^2) + \surd(1600+(240-x)^2)$, WITH X-VALUES SELECTED]
3.2 Using technology/mathematical tables to perform calculation [SUCCESSFUL CALCULATION OF L-VALUE]
3.3 Using technology to automate application of formulae to multiple cases [EFFECTIVE USE OF LIST FACILITY]
3.4 Using technology to produce graphical representations [SPREADSHEET CHART OR GC STATPLOT]
4. MATHEMATICAL SOLUTION → REAL WORLD MEANING OF SOLUTION
4.1 Identifying math results with real world counterparts [L-VALUES IN TERMS OF CHECKPOINT STATIONS]
4.2 Integrating arguments [OPTIMUM PLACEMENT OF STATION IN TERMS OF GRAPHICAL INTERPRETATION]
4.3 Relaxing *prior* constraints to address *new* situation [APPLYING NEW CRITERIA FOR 19TH STATION]
4.4 Invoking mathematics to support decision [NO OPTIMAL GUESS WITHOUT MATHEMATICAL SUPPORT]

Figure 4. Emergent framework for identifying student blockages in transitions.

Real world problem statement → mathematical model

Key requirements with the potential to generate blockages include the following: Selection of dependent and independent variables for distance formula (2.1); setting up a formula that uses *known* quantities (lengths along side of field) (2.2); realising that an independent variable must be uniquely defined (2.3) [e.g., Mei used a one variable expression for the total distance $\sqrt{(40^2 + x)} + \sqrt{(120^2 + x)}$, and (the obvious error apart) did not see conflict when using 'x' as the distance from the station to corner A in one part of the expression, and corner B in the other part]; choosing technologies to enable calculations such as square roots (e.g., with Excel versus a calculator) (2.4); choosing to use technology to automate extension of the application of formulae to multiple cases (using graphing calculator LISTs or spreadsheet) (2.5); choosing to use technology (spreadsheet or graphing calculator) to produce a graphical representation of the model (2.6).

Mathematical model → mathematical solution

In this transition students need the facility to carry through the strategic decisions made previously. Knowledge of mathematical procedures (3.1), technological knowledge for their automation, and declarative knowledge about the rules of notational syntax associated with both mathematics and technology feature in the sources of blockages in this transition (3.2, 3.3, 3.4). Some blockages here follow from difficulties in the earlier formulation process (e.g., non-uniqueness of a definition of a variable). Others follow appropriate decisions made in formulation (e.g., choosing to use technology for some correct purpose), but occur due to technical failures in using technology to automate extensions of formulae to multiple cases or to produce graphical representations. One student performed all 18 station related calculations by hand, at the cost of both time and experience in using technology to automate calculations already mastered by hand. She thus denied herself "reflective time" needed to examine the appropriateness and reasonableness of the models constructed in relation to real world aspects of the situation being modelled.

Mathematical solution → real world meaning of solution

Blockages occur as students fail to identify mathematical results with real world counterparts (4.1). This most basic of interpretative acts, involves here the interpretation of an outcome distance in terms of implications of using the corresponding checkpoint station, or the meaning of the minimum distance in terms of a particular strategy for station selection. The quality of interpretations ranged from bald assertions, to reasoned argument based on mathematical outcomes (4.2, 4.4). For example, when asked, "Does running via station 1, or station 2, or station 3 make any difference to the overall length of the run?" responses ranged from unsupported assertions such as, "It makes a difference", to justified arguments based on associated numerical results such as, "Yes, it does the closer you are to corner A, the further the distance you have to run." The ability to deal appropriately with constraints is another key skill and its absence a source of blockage (4.3). Students had the greatest difficulty determining where to place a 19th station, as this entailed relaxing the previous constraint of continuing the ordered pattern (19th must follow 18th at a distance of 10 metres). Many students simply placed the extra station 10 m away from either the first or last stations, rather than applying the minimum distance criterion.

Reflections

In conclusion we locate the current work within the wider field of applications and mathematical modelling in education. Firstly, a direct application derives from the way the research has been conducted. This is to identify specifically, activities with which modellers need to have competence in order to successfully apply mathematics at their level. The Framework is an attempt to begin to systematically document these. As the elements in the framework were identified by observing students working (and in particular wrestling with blockages to progress), there are two immediate potential applications. First are the insights obtained into student learning, and how these can inform our understanding of the ways that students act when faced with modelling problems. Second, closely allied to this are associated pedagogical insights. By identifying difficulties with generic properties, the possibility arises to anticipate where, in given problems, blockages of different types might be expected. This understanding can then contribute to the planning of teaching and task design, in particular the identification of prerequisite knowledge and skills, preparation for intervention at key points if required, and scaffolding of significant learning episodes. With respect to the different 'models' of modelling introduced earlier, the approach here provides some safeguards against the worst excesses of curve fitting elevated to a status beyond its importance. Designing problems where data are generated from a real context, before being subjected to analyses, reduces the potential for mindless manipulation driven by available calculator or computer menus.

There are design and implementation implications if modelling is used primarily to serve other curricular needs, rather than viewed as an ability to be built and nurtured in its own right. In the latter view, necessary knowledge and skills must first be available, as the emphasis is on using existing knowledge to solve problems with real world connections. Clearly cognitive demand is increased when attention must be divided between activating a modelling cycle and puzzling about technical aspects of technology (or indeed by-hand mathematics). This is not to say that students will not reach beyond their present level to involve mathematics new to them, or use technology in previously unexplored ways. The point is that the basic elements required to initiate and support such activity should be within their experience. In the present example this would include familiarity with spreadsheets, and facility with the use of LISTs and function graphing options on graphing calculators. In this approach to modelling student facility in transitions is a key, and independence here should be encouraged, scaffolded if necessary, (but not specifically led except as a last resort), to make and carry through essential decisions along a solution path. If however, modelling is used primarily to motivate, and/or provide a vehicle for the development of particular mathematical or technological expertise the situation is somewhat different. For example the calculation of distances for multiple stations could provide a context for the introduction and operation of the LIST facility. At the point where multiple calculations become necessary the modelling process would be interrupted, and teaching emphasis moved to mastery of a new skill. The provision of a meaningful context is the main purpose for the modelling in this view. Of course elements of both approaches can be incorporated in a given application. In *Cunning Running* an adequate solution at year 9 level can be obtained in terms of the spreadsheet graph in Figure 3 or by using the numerical representation (on the graphing calculator, spreadsheet or by hand). Introduction of the challenge to 'verify' the equation to the graph in this particular version of the task provides an opening for the introduction of the function graphing facility of a graphing calculator, which can then be pursued as a technique in itself.

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