Reframing Mathematical Futures: Using Learning Progressions to Support Mathematical Thinking in the Middle Years

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The Australian Curriculum: Mathematics calls for the concurrent development of mathematical skills and mathematical reasoning. What are the big ideas of mathematical reasoning and is it possible to map their learning trajectories? Using rich assessment tasks designed for middle-years students of mathematics, this symposium reports on the preliminary phase of a large national study designed to move beyond the hypothetical and to provide an evidence-based foundation for learning progressions in mathematical reasoning in three key areas of the curriculum: Algebraic Reasoning, Geometrical and Spatial Reasoning, and Statistical Reasoning.

Paper 1: Dianne Siemon. *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years – Introducing the Reframing Mathematical Futures II Project*

This paper presents an overview of the project and discusses the importance of mathematical reasoning.

Paper 2: Lorraine Day, Max Stephens, & Marj Horne. *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years – Algebraic Reasoning*

The results of the initial trialling of a set of items designed to identify algebraic reasoning, and the big ideas of algebra will be discussed.

Paper 3: Marj Horne & Rebecca Seah. *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years* – Geometric Reasoning

Little recent research addresses geometrical and spatial reasoning. This paper reports on a hypothesised learning hierarchy and the results from the trial process.

Paper 4: Jane Watson & Rosemary Callingham: *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years – Statistical Reasoning*

Using an existing research base, and the outcomes from trial tests, this paper describes a learning hierarchy of statistical reasoning.

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Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years: Introducing the Reframing Mathematical Futures II Project

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The Australian Curriculum: Mathematics calls for the concurrent development of mathematical skills and mathematical reasoning. What are the big ideas of mathematical reasoning and is it possible to map their learning trajectories? Using rich assessment tasks designed for middle-years students of mathematics, this paper reports on the preliminary phase of a large national study designed to move beyond the hypothetical and to provide an evidence-based foundation for learning progressions in mathematical reasoning in three key areas of the curriculum.

Why Mathematical Reasoning?

The Programme for International Student Assessment (PISA) results for 2012 and 2015 indicate a significant decline in mathematical literacy rates among Australian 15-year-olds since 2003 (Thomson, De Bortoli, & Buckley, 2013; Thomson, De Bortoli, & Underwood, 2016). In particular, the results reported in 2013 suggest that

> interpreting, applying and evaluating mathematical outcomes … is an area of relative strength for Australian students, while formulating situations mathematically and employing mathematical concepts, facts, procedures and reasoning are seemingly processes of relative weakness (p. x).

This is consistent with the Middle Years Numeracy Research Project (MYNRP), which found that many students in Years 5 to 9 experience considerable difficulty interpreting problem situations, applying what they know to solve unfamiliar situations, explaining their thinking, and communicating mathematically (Siemon, Virgona, & Corneille, 2001). It is also consistent with data from the Trends in International Mathematics and Science Study 1999 Video Study that led Stacey (2003) to call for an increased focus on mathematical reasoning. Although these capacities are recognised and valued in the Australian Curriculum: Mathematics (ACM) in the form of the four proficiencies, that is, conceptual understanding, procedural fluency, mathematical problem solving, and mathematical reasoning, these are often not reflected in school mathematics at this level where:

- mathematics is typically represented as a set of disconnected topics and skills to be demonstrated and practiced rather than explored, discussed and connected (Shields & Dole, 2013; Siemon, Bleckly, & Neal, 2012)
- the vast majority of textbook problems at Year 8 tend to be relatively low-level, skill-based repetitious exercises (Vincent & Stacey, 2008)

A focus on mathematical reasoning is needed to equip teachers with the knowledge, confidence and disposition to go beyond narrow skill-based approaches to teaching mathematics in the middle years. Defined broadly in the ACM as a "capacity for logical thought and actions", mathematical reasoning has a lot in common with mathematical problem solving, but it also relates to students' capacity to see beyond the particular to generalize and represent structural relationships, which is a key aspect of further study in mathematics and a key underpinning of Science Technology Engineering and Mathematics (STEM)-related studies (Wai, Lubinski, & Benbow, 2009).

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Why Learning Progressions?

Australian teachers of mathematics are familiar with scope and sequence charts and curriculum documents that imply broad developmental progressions in mathematics learning from the early to the post compulsory years of schooling. While the implied sequences in Number and Algebra are generally supported by research in the early years (e.g., Clarke et al., 2002; Mulligan & Mitchelmore, 2009), the evidence for the implied learning sequences beyond the early years and in other domains is less conclusive. One of the reasons for this is that although there has been considerable research on particular aspects of these domains in the middle years of schooling, much of this is “fragmented due to the variations in research questions and methods” (Confrey & Malone, 2014, p. xiv).

A more pervasive issue is the fact that curriculum content descriptors are generally expressed in a form that allows observation and measurement with little/no indication of their relative importance or how they connect to the ‘big ideas’ in mathematics needed to ensure students make progress. This situation inevitably privileges skills over concepts and de-emphasises the processes of mathematical problem solving and reasoning. Research is needed to identify big ideas and developmental pathways that underpin mathematical reasoning in the middle years of schooling to give “teachers, textbook authors and curriculum writers a sense of what type of reasoning they can expect and encourage at each level and in what directions students’ reasoning should be developed” (Stacey, 2010, p. 19).

It is only relatively recently that learning progressions/trajectories per se have become the focus of systematic research efforts (e.g., Clements, 2002; Confrey, 2008; Daro, Mosher, & Corcoran, 2011; Siemon, Izard, Breed, & Virgona, 2006). Prompted by Simon’s (1995) introduction of the notion of Hypothetical Learning Trajectories, there is debate about the meaning and use of learning progressions/trajectories in mathematics education (e.g., see the special edition of *Mathematics Teaching and Learning, 6*(2) in 2004). However, a common element in the different interpretations and use of the terms is the notion that learning takes place over time and that teaching involves recognising where learners are in their learning journey and providing challenging but achievable learning experiences that support learners’ progress to the next step in their particular journey.

Outline of the RMF II Project

Reframing Mathematical Futures II (RMFII) is a three-year project funded by the Australian Government Department of Education and Training under the auspices of the Australian Mathematics and Science Partnership Programme (AMSPP). The project is working with industry partners and practitioners in each State and Territory and the Australian Association of Mathematics Teachers (AAMT) to build a sustainable, evidence-based, integrated learning and teaching resource to support the development of mathematical reasoning in Years 7 to 10 comprising:

- evidence-based learning progressions in algebraic, statistical, and spatial reasoning that can be used to inform teaching decisions and the choice of mathematics learning activities and resources by teachers and students;
- a range of validated, rich assessment tasks and scoring rubrics that can be used to identify what students know and understand in terms of the learning progressions, inform starting points for teaching and show learning over time (i.e., as pre- and post-tests);

- detailed teaching advice linked to the learning progressions that establish and consolidate learning at the level identified and introduce and develop the ideas and strategies needed to progress learning to the next level of the framework;
- indicative resources to support the implementation of a targeted teaching approach in mixed ability classrooms.

Methodology of the RMF II Project

The RMFII project has been designed in terms of three distinct but overlapping phases. Phase 1 focussed on the identification of hypothetical learning progressions from the research literature to inform task design, the provision of professional learning to support a targeted teaching approach (Siemon, 2017) to mathematics in the middle years and the development and trial of rich tasks to assess algebraic, statistical and spatial reasoning at this level. Rasch modelling (Bond & Fox, 2015) was used to analyse data collected from the trial tasks and the findings used together with the hypothetical learning progressions to formulate Draft Learning Progressions in each area of interest.

Phase 2 is focussed on the preparation and use of multiple assessment forms for mathematical reasoning, the analysis of student and teacher surveys, and the development of teaching advice and professional learning modules to support a targeted teaching approach. The final phase of the project will focus on the development and publication of project outcomes and reports. Project partners in all Australian States and Territories identified between four to six secondary schools in their jurisdiction that met the funding requirements (i.e., located in lower socio-economic regions with diverse populations). A specialist teacher was identified from each school and is being supported by the research team to work with up to 6 other teachers in their school to trial assessment tasks and implement a targeted teaching approach to mathematical reasoning. A total of 32 secondary schools, approximately 80 teachers, and 3,500 students in Years 7 to 10 are involved in the project.

This symposium will consider preliminary findings from the first phase of the RMFII project, which was focussed on the development of evidence-based Draft Learning Progressions in algebraic, statistical and spatial reasoning. This phase was designed to address the following research questions.

- To what extent can we develop rich tasks to accurately identify key points in the development of mathematical reasoning in the junior secondary years?
- To what extent can we gather evidence about each student's achievements with respect to these key points to inform the development of a coherent learning and assessment framework for mathematical reasoning?

The first step in this process involved the derivation of hypothetical learning progressions in each domain from a review of the literature by specialist members of the research team. A range of assessment tasks and scoring rubrics were then devised to assess key elements of these progressions. These tasks were arranged in 24 different but overlapping forms and trialled with 3,075 students from 18 trial schools and coded by a team at RMIT University. The resulting data were analysed using the Rasch partial credit model (Masters, 1982) using Winsteps 3.92.0 (Linacre, 2016).

Rasch analysis allows both students' performances and item difficulties to be measured using the same log-odds unit (the logit), and placed on an interval scale (Bond & Fox, 2015). Items that did not fit the model were examined and refined. A small number of items was removed as not useful or too complex for students to understand. A refined set of overlapping forms was constructed and used with 3,366 students from participating research schools. This allowed the further refinement of the Draft Learning Progressions and it is these and detailed processes involved that are highlighted in this symposium.

The three related papers consider the derivation of the Draft Learning Progressions for algebraic reasoning, statistical reasoning, and geometric reasoning respectively. In each case, eight incremental Zones were identified on the basis of the hierarchy of items created by the Rasch analysis. Descriptors of student behaviour were derived from a consideration of the cognitive demands of items within each Zone. Where there are insufficient items in a Zone to address a particular 'big idea' or generate descriptors, additional items will be developed, trialled and used to further inform the Draft Learning Progressions.

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Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years: Algebraic Reasoning

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As part of the Reframing Mathematical Futures II Project on Mathematical Reasoning, algebraic reasoning was identified as one of the three areas to be investigated. This involved developing a hypothetical learning progression for algebra to inform the design of assessment tasks to test the progression. The assessment forms were then sent to trial schools and the data was analysed using Rasch Analysis. This paper reports on the analysis of the preliminary data received and outlines some implications for teaching.

The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2016) has combined Number and Algebra in a single strand to allow both to be developed together. Developing both numerical and algebraic reasoning together provides students with the opportunity to notice structure and powerful schemes for thinking about number patterns and relationships (Carpenter, Franke, & Levi, 2003). This implies that classroom practices need to adapt to build a more robust understanding of mathematics as a process of generalisation and formalisation, or as Kaput (1998) expressed it, 'algebrafying' the process. This transformation could be viewed as moving classroom practice from one of following rules and memorisation to one of sense-making (Flewelling, Kepner, & Ewing, 2007; Schoenfeld, 2008).

In order to identify a hypothetical learning progression for algebraic reasoning a review of the literature was conducted to identify the big ideas of algebra. Although the focus was to be on algebraic reasoning, it was considered appropriate to identify algebraic content, as students, at different levels, need content about which to reason. Underpinning this content focus was the understanding that in order to reason algebraically at the highest level involves visualisation, being able to move fluidly between multiple representations and having the language and discourse to reason mathematically.

Initially, hypothetical learning progressions were developed for five big ideas in algebra identified as: Pattern and Sequence, Generalisation, Function, Equivalence, and Equation Solving (Blanton, & Kaput, 2011; Blanton et al., 2015; Carraher, Schliemann, Brizuela, & Earnest, 2006; Fujii & Stephens, 2001; Mason, Stephens, & Watson, 2009; Panorkou, Maloney, & Confrey, 2013; Perso, 2003; Stephens & Armanto, 2010; Watson, 2009). However, as there was considerable overlap in the descriptors at this stage, it was decided to re-organise these in terms of: Pattern and Function, Equivalence, and Generalisation. An example of the hypothetical learning progression developed for Generalisation is shown in Table 1.

Table 1
The Hypothetical Learning Progression for Generalisation

Zone	Descriptor
1	Explain a generalisation of a simple physical situation.

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2	Explore and conjecture about patterns in the structure of number, identifying numbers that change and numbers that can vary.
3	Explain generalisations by telling stories in words, with materials and using symbols.
4	Explain generalisations using symbols and explore relationships using technology.
5	Follow, compare and explain rules for linking successive terms in a sequence or pair quantities using one or two operations.
6	Use and interpret basic algebraic conventions for representing situations involving a variable quantity.
7	Use and interpret algebraic conventions for representing generality and relationships between variables and establish equivalence using the distributive property and inverses of addition and multiplication.
8	Combine facility with symbolic representation and understanding of algebraic concepts to represent and explain mathematical situations.

Once the hypothetical learning progressions were identified on the basis of prior research, assessment tasks containing one or more items were compiled into forms that were designed to evaluate the three big ideas across Zones. Some tasks/items addressed a particular big idea while others assessed several of the big ideas in a single task. For instance, the seven-item Relational Thinking task was designed to evaluate key aspects of the hypothetical learning progressions for the two big ideas of Equivalence and Generalisation (see Table 2).

Table 2
The Relational Thinking Items and Rubrics

Item No.	Item		Rubric
1	What numbers would replace the ? to make a true number sentence (the numbers may be different). Explain your reasoning ? + 521 = 527 + ?	0	No response or irrelevant response
		1	Incorrect response but with correct reasoning based on the relationship between 521 and 527
		2	Two correct numbers given but little/no reasoning
		3	Two correct numbers given where the number on the left is 6 more than the number on the right with reasoning that reflects relationship between 521 and 527
2	Find a different pair of numbers that would make the number sentence above true	0	No response or irrelevant response
		1	A different and correct pair
3	Describe how you could find all possible pairs of numbers that would make this a true sentence.	0	No response or irrelevant response
		1	Incorrect attempt at describing based on previous answers
		2	Statement regarding difference of 6 or expression showing difference
4	What numbers would replace the ? to make a true number sentence (the numbers may be different)? ? – 521 = ? - 527	0	No response or irrelevant response
		1	Incorrect response but with correct reasoning based on the relationship between 521 and 527
		2	Two correct numbers given but little/no reasoning, may include some calculations
		3	Two correct numbers given where the number on the right is 6 more than the number on the left, with reasoning that

			reflects the relationship between 521 and 527
5	Find another set of numbers that would make the number sentence in 4 true.	0 1	No response or irrelevant response A different and correct pair
6	Describe how you could find all possible pairs of numbers that would make this a true number sentence	0 1 2	No response or irrelevant response Incorrect attempt at describing based on previous answers Statement regarding difference of 6 or expression showing the difference
7	What can you say about the relationship between *c* and *d* in this equation? $c \times 2 = d \times 14$	0	No response or irrelevant response Specific solution provided ($c = 7$ and $d = 1$) or a general statement (*c* is 7 times the number *d*) Statement correctly describes the relationship (*c* is 7 times the number *d*)

Results

Rasch analysis was used to rank student responses to the algebraic reasoning tasks and create a Draft Learning Progression for Algebra. From this it was possible to identify where different student responses to each of the Relational Thinking items were located on the progression. For instance, a score of 2 on RT1 (indicated by RT1.2 in Table 3 below) was located in Zone 3 while a score of 3 on RT1 (RT1.3) was located in Zone 6. Table 2 shows a range of responses to the RT items and their relationship to the big ideas of Equivalence (Equiv) or Generalisation (Gen).

Table 3
Results of Rasch analysis on the Relational Thinking Items

RT1.2	RT1.3	RT2.1	RT3.1	RT3.2	RT4.2	RT4.3	RT5.1	RT6.2	RT7.1	RT7.2
Zone 3	Zone 6	Zone 4	Zone 5	Zone 6	Zone 5	Zone 7	Zone 5	Zone 7	Zone 4	Zone 6
Equiv	Gen	Equiv	Equiv	Gen	Equiv	Gen	Equiv	Gen	Equiv	Gen

The different student responses indicated by the scores for each item in Table 2 range from Zone 3 to Zone 7. Those that relate to Equivalence range from Zone 3 to Zone 5. Finding a correct pair of numbers to make a correct number sentence (RT1.2) was the easiest at Zone 3. Finding another correct pair of numbers to the same question (RT2.1) was at Zone 4. Whereas, finding two correct pairs of numbers that satisfied the subtraction number sentence (RT4.4) was scaled higher at Zone 5. Components that required students to give a general explanation of a relationship were scaled at Zone 6 or Zone 7. Generalisation items were typically more difficult than Equivalence items; and among Generalisation items, as Table 2 shows, explanations involving subtraction or difference tended to be more difficult than those involving addition relationships. This confirms research findings by Stephens and Armanto (2010), Mason et al. (2009), and Carpenter et al. (2003).

In most cases incorrect responses to items in the Relational Thinking task were located in the lower Zones of the progression. For example, giving an incorrect response to the missing numbers in item 1 was scaled at Zone 1. However, an incorrect attempt at describing the relationship between the two missing numbers based on previous answers for item 2 was at Zone 4; and an incorrect attempt at describing the relationship based on previous answers for item 6 was scaled at Zone 5. These latter two results which embody incorrect or incomplete generalisations show that, for our upper primary and junior secondary students, generalisation and explanation of algebraic thinking remains quite difficult. As the research of Kaput et al. (1998), Carraher et al. (1996), and Blanton et al. (2015) demonstrated, helping students to articulate and refine their algebraic thinking, especially their algebraic reasoning and justification, are complex and challenging tasks even for capable teachers. These abilities require constant and supportive cultivation if they are to be achieved by most students. The preliminary data presented above show that they have been achieved by some students. Expanding the range of achievement, especially with respect to the development of reasoning, remains our challenge as this project moves into its next phase.

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Learning Progressions to Support Mathematical Reasoning in the Middle Years: Geometric Reasoning

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Geometric reasoning is an important construct in excelling in STEM related disciplines. Yet this is a topic that is most neglected and least understood by teachers and students alike. As part of the Reframing Mathematical Futures II Project, this paper reports the development of a learning progression that provides explicit validated mapping of students' growth in geometric thinking. Thirty-six items collated into two assessment forms were administered and analysed using Rasch modelling. Eight learning Zones were identified.

The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority, 2016) made a distinction between 'spatial reasoning', as one of the six interrelated elements in the numeracy learning continuum, and 'geometric reasoning' as one of the content descriptors to be taught from Year 3. Beyond this, there are scant details in clarifying the differences between these and how to help children develop both reasoning abilities. Spatial reasoning is the ability to see, inspect and reflect on spatial objects, images, relationships and transformations (Battista, 2007). Geometric reasoning is the use of critical thinking, and spatial reasoning to logically deduce and find new geometric relationships in problem situations. Its success is dependent upon the use of geometric knowledge and spatial reasoning to identify and formulate axiomatic relationships. Research affirmed that individuals progress through distinct stages of geometric thinking: visualise, describe, analyse, infer and deduce geometric relationships, and formal proof (Battista, 2007; van Hiele, 1986). These five stages are seen as interconnected and developed progressively with various degrees of emphasis and importance depending on the task demand. Proficiency in one domain is supported by good command of stages developed earlier. How to move students through these stages of thinking is the issue in question, hence the development of a hypothetical learning progression (HLP).

Learning progressions are a set of empirically grounded and testable hypotheses about how students' understanding of, and ability to use, specific discipline knowledge within a subject domain in increasingly sophisticated ways develops through appropriate instruction (Corcoran, Mosher, & Rogat, 2009). The purpose is to provide explicit validated mapping of students' growth in thinking and instructional advice on how to promote thinking through the stages. We take the position that the development of geometric thought is underpinned by the degree of connectedness between representation, visualisation and mathematical discourse. Geometric representations in the form of lines, shapes and diagrams are schematic, bound by their "formal concept definitions" (Tall & Vinner, 1981, p. 152) and developed through the process of visualising, a form of cognitive process in which objects are interpreted within the person's existing network of beliefs, experiences, and understanding (Phillips, Norris, & Macnab, 2010). How accurately individuals interpret the image is dependent upon how well they communicate what they see. Students literally talk themselves into understanding geometric properties. Evidence suggests that quality and targeted teaching is crucial to help shift thinking to the next level (Corcoran et al., 2009).

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Method

A survey of literature shows that inability to recognise geometrical shapes in non-standard orientation, perceive class inclusions of shapes, visualise geometrical solids in 2D format, and solve measurement problems that require spatial reasoning are well documented as problems students face when learning geometry (Battista, 2004; Burger & Shaughnessy, 1986; Elia & Gagatsis, 2003; Levenson, 2012). To ensure that sufficient data were collected to inform the design of the HLP, a bank of assessment questions was first designed and administered to test their reliability. The questions were designed to assess what we expect middle school students to be able to do, and focused on reasoning rather than procedural skills. They are grouped into three domains: (1) properties and hierarchy, (2) transformation of relationships, and (3) geometric measurement. Within each domain is a set of mathematical concepts vital for the development of geometric knowledge.

The first trial consisting of 62 items was administered to 390 students to determine its reliability and validity. These were marked by two markers and validated by a team of expert consultants to ensure the accuracy of the marking rubric and data entry. The questions were reduced to 36 items, collated into two forms and sent out to Year 7 to 9 students in participating schools. The test was administered to 755 students with 742 valid responses.

Results

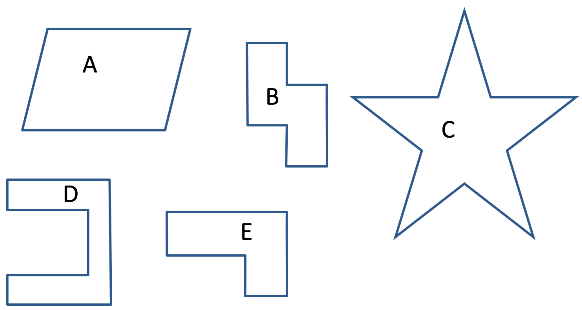
Rasch analysis of the responses resulted in identification of eight distinct thinking Zones (see Table 1). To facilitate better understanding of the formation of HLP, consider the concept of symmetry (Figure 1). It is an important aspect for developing spatial reasoning and understanding geometric properties as it promotes visualisation and geometric discourse as students learn to identify and reason about space and patterns. The code for each question, GSYM, indicates Geometry Symmetry.

5 Symmetry

Look at the shapes below

a [GSYM1] On each of these shapes draw all lines of symmetry.

b [GSYM2] For each of these shapes in part a, decide whether there is any reflectional or rotational symmetry and write the letters in the appropriate space in the table below.



	Has rotational symmetry	Does not have rotational symmetry
Has reflectional symmetry		
Does not have reflectional symmetry		

c [GSYM3] How do you know if a shape has rotational symmetry?

Figure 1. Assessment questions on concept of symmetry.

Score	Description for GSYM1	Description for GSYM2
0	No response or irrelevant response	No response or irrelevant response
1	No shapes having all lines drawn correctly	Only one shape (letter) correctly placed
2	All symmetry line(s) drawn correctly on one shape (others may be incorrect)	At least 2 correctly placed
3	Correct lines drawn on C and D but incorrect lines drawn on at least one other shape	At least 4 correctly placed
4	D: one line correctly drawn horizontally through centre C: five lines drawn from each point to opposite reflex angled corner No lines drawn on A, B or E	C has both rotational and reflectional symmetry A, B have rotational symmetry but no reflectional symmetry D has reflectional symmetry but no rotational symmetry E has no symmetry
	Description for GSYM3	
0	No response or irrelevant response	
1	An attempt at an explanation but lacking clarity	
2	Some explanation about turning shape part way around circle and it looking the same – perhaps around a pin	

Figure 2. Marking rubric for the symmetry question.

The eight Zones that were identified by the Rasch analysis are shown in Table 1. The responses to the symmetry question ranged from Zone 2 to Zone 8. Completing GSYM1 and GSYM2 with a score of 4 was in Zone 8 demonstrating understanding of reflectional and rotational symmetry. Giving a clear explanation of rotational symmetry in GSYM3 was in Zone 6. For GSYM2, students correctly placing one of the shapes in the correct position (score 1) demonstrated that they could visualise the 2D shape from a different perspective, rotating it and reflecting it, hence Zone 4, whereas identifying the reflectional symmetry of one shape (GSYM1, score 1) was in Zone 2.

Table 1
Hypothetical Learning Progression for Geometric Reasoning

Zone	Indicators
1	Recognises shapes by appearance and common orientation, shows emerging recognition of objects from different perspective, symmetry of objects and shapes and coordinate system.
2	Recognises reflection symmetry, nets of simple solids and simple shapes, shows emerging understanding of measurement concepts.
3	Able to visualise some objects from different perspective and use coordinates, uses one or two properties or attributes (insufficient) to explain their reasoning about shapes and measurement.
4	Able to visualise objects from different perspective, incomplete reasoning in geometric and measurement situations, performs measurement calculations but attends to only one attribute, gives directions on a map from personal rather than other viewer's perspective.

5	Able to visualise and represent 3D objects using 2D platform (Nets), uses either properties or orientations to reason in geometric situation, uses landmarks but retains personal orientation when providing directions, demonstrates knowledge of dilution and coordinate systems, provides partial solutions and explanations when calculating measurement situations.
6	Able to make deductions about angle situations and use properties accurately when reasoning about spatial situations but explanations are limited and lack knowledge of geometry hierarchy, provides accurate directions from a map, geometric and measurement arguments rely on examples/counter examples, omits one step when calculating multi-step measurement problems.
7	Beginning to recognise necessary and sufficient conditions, uses sound reasoning in argument/explanations, explanations often are procedurally based.
8	Constructs arguments based on multiple properties of 2D shapes and 3D objects, using the necessary and sufficient conditions to reason about geometric and measurement situations, conjectures and propositions (and theorems); demonstrates understanding of both reflectional and rotational symmetry.

These Zones reflect current students' geometric reasoning. Australian students compare poorly with students from other countries in geometry (Thompson, 2010). These Zones are not as advanced as we would desire for our students in years 7-10. Further research on improving teaching and learning practices will help to refine the HLP (Briggs & Peck, 2015). Indeed, the challenge now is to use this information to assist teachers to improve geometric reasoning in their classrooms and this assessment provides a tool to assist teachers to focus on targeting their teaching to the key ideas necessary in the development of understanding and reasoning in geometry.

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Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years: Statistical Reasoning

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As part of the Reframing Mathematical Futures II (RMFII) Project, Statistical Reasoning was identified as one of three areas of Mathematical Reasoning to be investigated. A hypothetical learning progression for statistics was developed based on previous research. Assessment tasks designed to address different Zones of the progression were sent to trial schools and the data were analysed using Rasch Analysis. This paper reports on the analysis of the preliminary data received and gives some implications for teaching.

Statistical reasoning has been a more recent addition to the mathematics curriculum than algebraic or geometric reasoning. Following the National Council of Teachers of Mathematics (NCTM, 1989) in the United States, *A National Statement on Mathematics for Australian Schools* was published by the Australian Education Council (AEC) in 1991. 'Chance and Data' was one of five content areas covered in four Bands (A to D) over the years of schooling. The expectations of the *National Statement* were the foundation for much of the research in statistics education at the school level in Australia in the 1990s, conducted using surveys and interviews with students from Year 3 to Year 10. Basing analysis of student responses on the SOLO Model of Biggs and Collis (1982), hierarchical development was identified in relation, for example, to beginning inference (Watson & Moritz, 1999), to sampling (Watson & Moritz, 2000a), to average (Watson & Moritz, 2000b), and to probabilistic beliefs (Watson & Moritz, 2003). This work was consolidated for survey data by Watson and Callingham (2003), in a suggested six-level uni-dimensional construct for statistical literacy. Further analysis by Callingham and Watson (2005) identified three subgroups of items that were related to the Chance and Data content of the curriculum. These were named Average/Chance (AC), Sample/Inference (SI), and Graphing/Variation (GV).

In the light of the more recent release of the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority, 2010-2016), these three subgroups were considered against the five statistics and probability big ideas proposed as a foundation for the mathematics curriculum implementation (Watson, Fitzallen, & Carter, 2013): Variation, Expectation, Distribution, Randomness, and Informal Inference. Variation is the fundamental concept underlying the others, Expectation underpins chance and calculations of averages, Informal Inference and Randomness cover appreciation of sampling, and Distribution includes graphing as well as other representations. Recognising the fundamental influence of Variation, the three reasoning big ideas for the Reframing Mathematical Futures II (RMFII) project were hypothesised as Variation in Expectation (AC), Variation in Distribution (GV), and Variation in Inference (SI). These big ideas can be considered separately at the beginning of a learning progression but, as learning progresses, they interact with each other to provide more sophisticated reasoning. Figure 1 shows the hypothetical learning progression for statistical reasoning, including eight Zones to be consistent with the previous results related to multiplicative thinking (Siemon, Virgona, & Corneille, 2001).

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Big Idea / Zone	Expectation (AC)	Distribution (GV)	Inference (SI)
Zone 1	**Idiosyncratic response or single procedural focus**		
	Uncertainty expressed as 50%	Reads single value on graph	Ignores context
Zone 2	**Considers aggregated information but without recognising value**		
	Anything can happen	Describes isolated features of a graph	One characteristic of a sample
Zone 3	**Emerging statistical appreciation but without explanation**		
	Claims for average without justification	Elaborated physical description of graphs	Choose "all" for sample
Zone 4	**Recognises influence of variation but interprets inappropriately**		
	Rejects "luck"; suggests unlikely	Does not distinguish scale in graph reading	Recognises sample but not its bias
Zone 5	**Straightforward explanation and simple numerical justification**		
	Orders chance phrases correctly	Appropriate attention to graph details	Partial recognition of sample requirements
Zone 6	**Informal appreciation of uncertainty and variation in choices**		
	Recognises outlier	Recognises correct variation in graphs	Suggests random sampling
Zone 7	**Makes inferences across ideas using proportional reasoning**		
	Creates appropriate probability distribution	Creates hypothesis based on data	Criticises sample size and bias
Zone 8	**Integrates proportional, statistical, and contextual reasoning**		
	Correct association in 2-way tables	Conclusion with both positives & negatives	Includes human/ psychological component

Figure 1. Hypothetical learning progression for statistical reasoning with selected examples.

Based on the hypothetical learning progression for statistical reasoning in Figure 1 and items used in previous research, assessment forms were devised for the middle school students in the RMFII project consisting of statistical reasoning tasks each of which had one or more items. Three forms included only statistics tasks, whereas six others contained a mix of statistics tasks with other tasks from algebra and geometry. Common tasks and items linked the forms. The rubrics for individual items suggested scores of between two and five to distinguish increasingly sophisticated responses. The Rasch analysis allocated the rubric scores across eight Zones of the construct of statistical reasoning, mapping students' overall performances on a logit scale in relation to the difficulty of the items.

Results and Discussion

No single task addressed every Zone of the hypothetical learning progression for statistical reasoning, but when considered together some tasks with their associated items related to the same context within a big idea and provided results across several Zones. An example shown in Figure 2 employs a task related to the tossing of a fair coin with four items. Items were labelled using the convention S for statistical reasoning with a 3 letter identifier that described the context (CON for coin toss) with a number/letter identifier for the specific item. The analysis of item difficulty for the items in Figure 2 illustrates each of the Zones in the hypothetical learning progression for statistical reasoning. At Zone 1, SCON1A and SCON1B show idiosyncratic reasoning, a response being "I think 2 tails…because 4÷2=2 so the average is 2" for SCON1B. Zone 2 is the highest response for SCON1A but more sophistication is possible for SCON1B with Zone 2 responses including "it's a 50% chance" or "you can't really tell." The lowest level of SCON2 is

Zone 3, with responses indicating equally likely proportions (e.g., “20, 20, 20, 20, 20”) or apparently random proportions (e.g., “10, 30, 40, 1, 19”).

SCON1A: Imagine you are playing game where you throw a coin 4 times. How many tails do you think might come up?

Score	Zone	Rubric Description for SCON1A
0		No response or irrelevant response
1	1	Any other number, “you don’t know, could be any of them”.
2	2	2 tails or 50%

SCON1B: Explain why.

Score	Zone	Rubric Description for SCON1B
0		No response or irrelevant response
1	1	Idiosyncratic reasoning or possible misinterpretation.
2	2	2 because there is a 50% chance, 50-50 of throwing a head or tail, probability of a tail 1 in 2.
		Recognises variation by stating “You can’t really tell how many tails might come up”.
3	8	2 but also recognising variation and/or attempts to quantify the highest likelihood.

SCON2: Imagine you are playing a game where you throw a coin 4 times. Imagine that 100 people played the game. In the table below, fill in how many people you think will get each number of tails.

Number of tails	Number of people getting the number of tails
0	
1	
2	
3	
4	
Total	100

Score	Zone	Rubric Description for SCON2
0		No response or irrelevant response, or does not add to 100.
1	3	Assumes equality for all options.
		Seemingly random prediction; gets proportion in the wrong spot.
2	6	Too narrow or no variation – extreme probabilistic outcome.
		Primitive understanding of proportion – 50% chance for 2 tails.
3	7	Appropriate variability displayed incorporating probability and distribution.

SCON3: Explain why you think these numbers are reasonable.

Score	Zone	Rubric Description for SCON2
0		No response or irrelevant response.
1	4	Idiosyncratic reasoning and personal beliefs.
2	5	Reasoning reflecting an even or equal chance for all numbers..
		Anything can happen, chance and luck.
3	6	Implicit understanding of chance and probability, sometimes mentioning 50%, or ½ chance (or answer reflecting a ‘4’ but not as clear).
4	8	Reasoning reflecting aspects of chance and probability including some kind of variability.

Figure 2. Example of a task having four items with scores across the hypothetical learning progression for statistical reasoning.

Score 1 in Zone 4 for SCON3, an explanation, reverts to personal beliefs in explaining the choices, but within a much more complex context. Whereas choice of equality of outcomes is Zone 3 for SCON2, for SCON3 it is Zone 5 reflecting the greater complexity, e.g., “because it adds up to 100.” Zone 6 shows the first primitive use of proportion, e.g., in choosing “5, 20, 50, 20, 5” for SCON2, and explaining the values for 100 tosses of a coin four times for SCON3, such as “it is more likely people will get 2 out of 4.” For SCON2, a Zone 7 response incudes variability within reasonable limits for the appropriate proportions. Finally explicit recognition of variation characterises responses at Zone 8. In explaining the original likelihood of two tails in four tosses, a response for SCON1B might be “2, because it has a 37.5% chance but the others could happen, they are just less chance.” For top responses to SCON3, again explicit mention of variability or likelihood is included in the explanation. The progression of difficulty between the first two items (SCON1A, SCON1B) and second two items (SCON2, SCON3) illustrates the contribution of context as the items move from single outcomes to multiple outcomes.

Several years will pass as students develop the understanding, first to justify the actual probability for obtaining two heads when tossing a coin four times and appreciate the variation associated with experimentally checking the value, and second to move to imagining 100 such repetitions of the four tosses and experiencing variation on completing trials. After completing hands-on activities, this problem presents an excellent opportunity to introduce computer simulations in the classroom, comparing outcomes among students, and increasing the number of trials beyond what can be done by hand.

The aim of the RMFII project is to provide teachers with ways of identifying and acting upon their students’ demonstrated reasoning, in this instance in statistical contexts. The initial findings indicate that it is possible to identify a progression and to match tasks and items to Zones of this progression. The next phase is to develop materials to lead students to being better able to explain their thinking and reasoning as they problem solve.

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