Developing early Place-value Understanding: A Framework for Tens Awareness

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This paper outlines a framework to explain the early development of place-value understanding based on an analysis of data from 84 five- to seven-year-old children from diverse cultural and linguistic backgrounds. The children were assessed individually on number knowledge tasks (recalled facts, subitizing, counting, place-value understanding) and strategies for solving word problems (addition, subtraction, multiplication, division). Children were categorised as working at one of four levels, each reflecting an increasing awareness of the structure of place value.

Background

Over the past few decades, the mathematics reform agenda has led to the revision of mathematics curricula worldwide. Curriculum focus for young learners usually starts with addition, subtraction, and place value, with multiplication and division introduced some years later. The expectation that children should know place value before experiencing multiplication and division is typical of various education systems (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2013; Common Core State Standards Initiative [CCSSI], 2010; Department for Education, 2013; Ministry of Education, 2007). This expectation overlooks the fact that multiplication and division provide important conceptual foundations for place-value understanding (Askew, 2013; Bakker, van den Heuval-Panhuizen, & Robitzch, 2014; Carpenter, Fennema, Franke, Levi, & Empson, 2015; Nunes et al., 2009; Ross, 1989).

Place-value understanding is clearly influenced by language factors such as the transparency of structure for units of ten and one. The irregularities and inconsistencies in the English language (e.g., '-teen' & '-ty' numbers) contrast with the transparent patterns found in most Asian languages. Research shows more advanced place-value understanding in Asian children than English speakers (Miura, Okamoto, Kim, Steere, & Fayol, 1993).

An important idea underpinning mathematics learning is the concept of "unit" (Behr, Harel, Post, & Lesh, 1994; Sophian, 2007). There is an assumption in early mathematics that "all quantities are represented in terms of units of one" (Behr et al., 1994, p. 123). It has been argued that learners should experience units other than one from the start of school to help them become aware of the usefulness of working with groups (Behr et al., 1994; Sophian, 2007). Evidence shows a small but persistent group of low-achieving students who continue to use counting by ones right through to adolescence, and this seriously limits their understanding of formal algebra (Young-Loveridge, 2010).

An alternative to counting strategies is decomposition of a quantity into parts (e.g., subtraction & division), and composition of parts to form a new whole (e.g., addition & multiplication) known as part-whole relations (Baroody, 2004). Place-value understanding provides an example of the need for part-whole thinking, with multi-digit quantities composed of units of increasing powers of ten, according to the position of each digit within the numeral representing the quantity (Ross, 1989).

The distinction between counting and part-whole strategies is reflected in two different conceptions of number: a *counting*-based (sequence-focused) conception and a *collections*-

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based (groups-focused) conception of number (Yackel, 2001). When place-value instruction begins, the emphasis needs to be on collections: groups of ten and groups of one. Yang and Cobb (1995) contrast the collections-based approach of Chinese mothers and teachers who emphasize groups (units) of ten, with the way that Western children are initially encouraged to count by ones (unitary). The more advanced place-value understanding by Chinese children compared to those in the US may be explained by this difference in emphasis on grouping by tens. A shift from unitary (by ones) to multi-unit conceptions of number is a characteristic of some developmental models of place-value understanding (e.g., Fuson, et al., 1997b). Researchers have shown that with carefully planned learning experiences, first grade students can learn the beginnings of place-value structure (e.g., Kari & Anderson, 2003).

Theoretical Framework

The theoretical framework proposed here is based on recent work outlining learning progressions in children's mathematical thinking (Clements & Sarama, 2014; Confrey, Maloney, & Nguyen, 2014; Weber, Walkington, & McGalliard, 2015). It is also informed by research on children's Awareness of Mathematical Pattern and Structure (AMPS: Mulligan & Mitchmore, 2009; Mulligan, 2011; Mulligan, Mitchelmore, & Stephanou, 2015). Mulligan and her colleagues showed that low-achieving students had limited AMPS and tended to use counting by ones to solve problems rather than flexible mental strategies.

The original study from which the data in the present study was abstracted set out to explore the impact of using multiplication and division contexts with five- to seven-year-olds on their number knowledge (including emerging place-value knowledge) and understanding of number operations. The study found that children as young as five years of age were able to solve multiplication and division problems providing they were presented in familiar contexts. Moreover, quotitive division into groups of ten (with remainder) was found to be helpful in supporting students' understanding of place-value. The data were analysed to investigate the question of how mathematical knowledge and concepts relevant to place-value understanding are acquired, and whether it is possible to identify an hierarchical progression of ideas (learning trajectory) that might have implications for instruction.

The Study

The study that informed the development of this theoretical framework was set in an urban school (medium SES) in New Zealand and included four classes (two Year 1, and a class each at Year 2 and Year 3). The participants were 84 Years 1 to 3 students; 35 five-year-olds (Y1), 24 six-year-olds (Y2), and 25 seven-year-olds (Y3). At the beginning of the study the students' ages ranged from 5.0 to 7.9 years (average: 6.3). One-third of the children were European, one-third were Māori (the indigenous people of NZ), and the remaining third consisted of Pasifika, Asian, and African ethnicities. One-fifth of the children were identified as English Language Learners [ELL].

Initially, the children were assessed individually using a diagnostic task-based interview designed to explore their number knowledge and problem-solving strategies (March). The children in the study were reassessed after a series of focused lessons during two four-week teaching blocks in May and August (September). The assessment tasks included: addition, subtraction, multiplication, division, known facts, incrementing in tens, counting sequences, subitizing, and place value.

Each lesson in the study began with all students completing a problem together, using materials to support the modelling process, and sharing ways of finding a solution. The problem for the day was written in a large scrapbook. As the children solved the problem, the teacher recorded their problem-solving strategies and solutions, acknowledging individual children's contributions to discussion. Both drawings and number sentences were recorded, with the teacher helping children to make connections between multiplication and division, two-digit numbers, and place value. The children then completed a problem in their own project books, choosing a similar or larger number, and/or self-selecting a new number. The children were encouraged to show their thinking in their project books using drawings with matching equations.

In the first series of lessons the children were introduced to multiplication with groups of two and five using familiar contexts such as pairs of socks and five candles on a birthday cake. Problems involving quotitive division used similar contexts. When solving problems with divisor two, odd numbers were deliberately used to introduce the idea of a remainder, as a precursor to quotitive division into groups of ten and "leftover" ones:

There are 13 mittens in the basket. How many pairs can we make?

Multiplication problems using groups of ten were then introduced using the contexts of eggs in cartons and chocolates in trays. This was followed by quotitive division into groups of ten using the context of filling egg cartons that held exactly ten eggs. A typical problem was:

There are 48 eggs. Each carton holds 10 eggs. How many full cartons are there?

Results

Tasks related to place value were selected, including joining or splitting quantities into groups of five and ten, incrementing or decrementing by tens. The tasks were then arranged in order of difficulty and grouped according to logical connections between the concepts being assessed. The 84 children were divided into four groups according to their performance on selected tasks (see Table 1). Highlighting is used to show the groups of children who had mastered the tasks at each level, using 80 per cent accuracy as the criterion for mastery.

At the lowest level, (Group 1) consisted of children (n=19) who were still learning to recall the facts for 5+5 and 10+10, and were not successful on the five tasks at Level 1. Less than 80 per cent of children in the group were successful on the easiest task (5+5). Few of these children were successful on tasks judged to be at higher levels on the proposed framework. Group 2 (n=20) had mastered the five tasks assessing recall of 5+5 and 10+10 facts, but were still learning about how to combine tens and ones, presented either as facts (e.g., 20+7) or as ten-frames and singleton dots in the form of a subitizing task (Note: only those children who were able to identify the quantity without counting were credited with success on these tasks). Group 3 (n=23) had mastered the recall of 5+5 and 10+10, as well as mostly knowing combinations of tens and ones presented either as facts or as part of subitizing ten-frames. This group was learning about multiplication and division with groups (units) of ten and ones, as well as increasing or decreasing a two-digit number by ten. Group 4 (n=22) had mastered all of the tasks described so far, but were learning about multiplication and division with groups of ten for three-digit numbers (e.g., how many $10 notes in $240), and increasing or decreasing a three-digit number by ten (e.g., ten more than 139).

Table 1
Percentages of children in each group who were successful on each task (highlighting indicates mastery of the task at 80 per cent)

Level	*Task*	Group 1 n=19	Group 2 n=20	Group 3 n=23	Group 4 n=22	*Total n=84*
1. Know facts for 2 gps of five & ten	a. Know 5+5 as a Fact	79	100	100	100	*95*
	b. Subitize 2 dice patterns of 5	68	100	100	100	*93*
	c. Subitize 1 full ten-frame	58	95	100	100	*89*
	d. Know 10+10 as a Fact	42	95	100	100	*86*
	e. Subitize 2 full ten-frames	16	95	100	100	*80*
2. Know tens plus ones	a. Subitize 1 ten-frame, 3 dots	5	50	96	100	*65*
	b. Subitize 2 ten-frames, 3 dots	0	45	91	100	*62*
	c. Know 20+7 as a Fact	11	25	87	100	*58*
	d. Know 10+8 as a Fact	5	20	83	100	*55*
3. Know mult/div w tens & add/sub ten to/from 2-digit nos.	a. Make $31: $10 notes, $1coins	0	10	61	100	*45*
	b. Know half of 20	0	15	74	95	*49*
	c. Know groups of 10 in '23'	5	10	39	95	*39*
	d. Know ten less than '20'	0	0	22	95	*31*
	e. Link '2' in '24' to groups of 10	0	10	52	91	*40*
	f. Know $10 notes in '$80'	5	5	30	91	*35*
	g. Know groups of 10 in '60'	0	5	13	91	*29*
	h. Know ten more than '19'	0	0	13	82	*25*
4. Know mult/div w tens & add/sub ten to/from 3-digit nos.	a. Know ten less than '100'	0	0	17	77	*25*
	b. Know ten more than '99'	0	0	9	64	*19*
	c. Know ten more than '139'	0	0	0	55	*25*
	d. Know $10s in '$240'	0	0	9	45	*14*
	e. Know ten more than '899'	0	0	0	41	*11*
	f. Know ten less than '600'	0	0	0	36	*10*
	g. Know $100 in $1600	0	0	4	23	*7*

Interestingly, there were children from each year level in each of the four groups, with one Year 3 child in Group 1, and one Year 1 child in Group 4. However, overall, Groups 1 and 2 consisted of younger students, while Groups 3 and 4 tended to be older students. English language proficiency did not appear to be an important issue as Groups 3 and 4 included the greatest proportion of English language learners.

As well as tasks assessing students' awareness of ten-structure, problems were given for each of the operations focusing on the nature of the strategy used to solve the problem, and tasks to assess knowledge of number sequence such as counting by ones and by multiples (see Table 2).

It is evident from Table 2 that very few of the students in either Group 1 or 2 were able to use part-whole strategies to solve problems. It was interesting to note that working with groups of 5 (in 4x5) was easier than either groups of 2 (6x2) or groups of 10 (3x10). It should be noted that in the 6x2 task, the objects (shells) were inside woven flax bags and hence screened from the student's view.

Students in both Groups 1 and 2 were both still consolidating their knowledge of sequence to 100 by ones, with less than one-third of students having mastered this sequence. Stopping points for Groups 1 and 2 included 15 (n=2), 20, 29, 36 (n=1 each), 39 (n=4), and 49 (n=3). Students in Groups 3 and 4 had completely mastered the sequence to 100 by ones, fives, and tens. For Group 3, the challenge was in learning to count by twos.

Table 2
Percentage of students in each group who used part-whole strategies to solve problems and produced counting sequences to at least 100 (Q=Quotitive division)

	Group				
Task	1	2	3	4	*Total*
	n=19	n=20	n=23	n=22	*n=84*
Addition/Subtraction Problems					
3+4 (screened)	5	10	17	59	*24*
23+4 (screened)	0	10	30	73	*30*
5+8 (screened)	0	0	4	41	*12*
42+30 (screened)	0	5	0	64	*18*
14-5 (screened)	0	0	0	50	*13*
37-9 (screened)	0	0	0	14	*4*
Multiplication					
6x2 (screened)	0	0	4	36	*11*
4x5 (pictured)	0	10	43	91	*38*
3x10 (pictured)	0	0	13	59	*19*
half of 100 (imagined)	0	5	39	100	*38*
Counting to 100					
by ones	26	30	100	100	*67*
by twos	5	10	52	95	*43*
by fives	16	35	96	100	*64*
by tens	26	50	96	95	*69*

Discussion

The analysis of selected data from this study was consistent with research on learning trajectories in mathematics showing hierarchical progression in the acquisition of concepts from simple through to relatively sophisticated understanding (e.g., Clements & Sarama, 2014; Confrey, et al., 2014). The students in Group 4 who were working on some of the most challenging place-value tasks and concepts were able to do all of the easier tasks. Those in Group 1 were not consistently successful with the easiest tasks, and were unable to complete harder tasks satisfactorily.

Interestingly, the learning trajectory proposed by Confrey et al. (2009) focuses exclusively on partitive division (equipartitioning/splitting) as being foundational for rational number reasoning. There is no mention by Confrey and her colleagues of quotitive division and its possible role in supporting students' understanding of rational number concepts. Place-value understanding with multi-digit whole numbers depends on understanding how quantities can be partitioned into units that are powers of ten (hundreds, tens, ones, etc.). Quotitive division into groups of ten, with explicit links to the digit representing the number of units of a particular denomination, provides a powerful opportunity to understand place value more deeply. Place-value understanding with decimal quantities simply extends that knowledge downwards to units that are smaller than one (tenths, hundredths, thousandths, etc.). When the unit size is already specified, as it is in the decimal system, it is quotitive, not partitive division, that helps make sense of a problem such as 'How many 0.3 kilo meat patties can be made from 2.4 kilos of minced meat?'

The children's performance on the tasks used in the study showed that the quinary and decade structure was helpful in identifying groups of ten as represented on a ten-frame by two rows of five dots (Task 2e). These findings are consistent with the idea that awareness of mathematical pattern and structure is important in children's mathematics learning (Mulligan, 2011; Mulligan & Mitchelmore, 2009; Mulligan et al., 2015; Wright, Ellemor-Collins, & Tabor, 2012). Although the majority of children could identify the quantity in two full ten-frames (80%), a combination of full ten-frames and singleton dots was considerably harder (Task 2a: 65% and Task 2b: 62%). It was children in Group 4, who had mastered the combining of tens with ones in the context of ten-frames or when presented simply as a number fact who were able to show how the '2' in '24' means two sticks of ten blocks. They were also able to identify how many \$10 notes would be needed to buy something costing \$80 (Task 3f).

The findings of study show that even children as young as five years of age were able to work with multiplication and division problems with two-digit numbers, and early place-value tasks. The word problems involved composing a new total using equal-sized groups (multiplication), and decomposing a quantity into groups of a particular size (quotitive division) with and without remainder. The use of familiar contexts and materials helped the children to appreciate this 'groups of' idea. The study showed that progressing from groups of two to groups of five, followed by groups of ten, led easily to the concept of units of ten and units of one as an inherent part of place-value understanding. The children were able to appreciate the structure of two-digit numbers by making complete sets of ten using ten-frames, and having some 'leftovers'' represented by the 'ones' digit. Despite the focus on 'groups of' and working with larger numbers than traditionally used with five- to seven-year-olds, the children's addition and subtraction strategies still tended to use counting on by ones rather than knowledge of part-whole relationships.

Children showed evidence of derived-fact strategies by using 'five plus five make ten' or 'two fives make ten' as a way to solve some multiplication problems. This use of 'two fives make ten' further supports the foundations for place-value understanding, and is consistent with the idea of introducing place value in the early years (e.g., Fuson, Smith, & Cicero, 1997a; Kari & Anderson, 2003). The findings raise questions about national and international curricula that introduce place value before multiplication and division (e.g., ACARA, 2013; CCSSI, 2010; Department for Education, 2007; Ministry of Education, 2007).

The superior performance of the children on 5+5 and 10+10 (95% & 86% accuracy) compared to that for sums with totals up to five (2+3: 52%; 1+4: 65%) challenge curriculum documents that expect facts with sums up to five to be learned before facts with larger sums (see Ministry of Education, 2007). These findings suggest that benchmark facts such as 5+5 and 10+10 need to be acknowledged as important for learning about place value and given priority over other facts. The salience of fingers on hands and toes on feet can be used to help consolidate students' knowledge of these important facts.

The opportunity for children to work with word problems involving multiplication and division contexts and having access to large 2-digit numbers enabled these children to make substantial progress in mathematics. Not only did these experiences enhance their understanding of multiplication and division, but they also helped to lay down foundations for place-value understanding. The emphasis on groups of two, five, and ten is consistent with the emphasis in the literature on the importance of moving away from unitary counting-based strategies to collections-based (i.e., groups of tens and groups of ones) strategies in order to build place-value understanding (Fuson et al., 1997a, 1997b; Yang & Cobb, 1995).

The framework proposed in this paper has important implications for practitioners. It could help teachers to think about students' learning of place value as a developmental progression with key features to guide their instructional decisions. The initial emphasis on learning facts for 5+5 and 10+10 provides an important building block on which the more sophisticated aspects of place value rest. Learning these two facts appears to be far more important than focusing on number facts for smaller quantities, contrary to current curriculum guidelines (e.g., Ministry of Education, 2007). Another key element is learning how multiples of ten can be combined with single-digit quantities so that counting processes are not necessary to determine the total quantity. The next step is to make explicit links between digits in a two-digit numeral and the groups of ten and 'leftover' ones that result from quotitive division. This understanding can be further supported by exploring what happens to a quantity, and to the numeral representing this quantity, when a group of ten is added or taken away from that quantity. A follow-up study to investigate the impact on students' learning of teachers using this framework could be extremely valuable.

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