High school students’ knowledge of a square as a basis for developing a geometric learning progression

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This study surveyed and analysed four secondary school students’ writing about a square. Sfard’s discursive approach to understanding mathematical discourse was used to analyse the responses collected from 214 Australian secondary school students. The results showed that geometric knowledge was developed experientially and not developmentally. This in turn helps refining the development of a geometric learning progression, with the accompaniment of a set of validated assessment tools and learning tasks that seeks to deepen teachers’ understanding of geometric reasoning and support student learning.

Spatial reasoning plays a pivotal role in succeeding in Science, Technology, Engineering and Mathematics disciplines (Wai, Lubinski, Benbow, 2009). Research indicates that many students and teachers have difficulties reasoning spatially, including an inability to: (1) recognise geometrical shapes in non-standard orientation, (2) perceive class inclusions of shapes, (3) visualise geometrical solids in 2D format, and (4) solve problems that require spatial reasoning (Elia & Gagatsis, 2003; Marchis, 2012). A lack of spatial and geometric reasoning ability can inhibit individuals from engaging in tasks that requires visual, logical and deductive thought.

Funded by the Department of Industry, Innovation, Climate Change, Science, Research and Tertiary Education, this research project, *Reframing Mathematical Futures II,* aims to develop, trial and evaluate a set of validated assessment tools and learning tasks that will deepen teachers’ understanding of mathematical (multiplicative, algebraic, geometric, and statistical) reasoning and support student learning. While students’ difficulties with geometry have been documented elsewhere, data on current Australian students’ geometry knowledge is lacking. The absence of emphasis on spatial reasoning in the Australian national curriculum would lead one to question the state of geometry teaching in Australian schools. Accordingly, a survey of students’ geometric knowledge is needed to inform the formation of a learning progression, which can then be used to guide the development, trialling and evaluation of a set of validated assessment tools and classroom tasks to support student learning.

Theoretical Framework

The Learning Progression for Geometry

The learning progression formulated for this study is based on the premise that all types of geometric concepts develop over time, becoming increasingly integrated and synthesised (Jones, 2002). The term learning progression should not be confused with ‘developmental progression’ used in the hypothetical learning trajectory model (Clements, Wilson & Sarama, 2004). For Clements et al., a developmental progression is constructed with a learning goal in mind and supported by specific instructional sequence designed to promote key learning tasks. The proposed learning progression is a mapping of the

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development of geometric knowledge and its corresponding measurement ideas. A discussion on learning two-dimensional shapes (2D) is addressed here.

Acknowledging the work of Battista (2007) and van Hiele (1986), this learning progression views the development of geometric knowledge as moving through five levels of reasoning: visualising, describing, analysing, and inferring geometric relationships, leading to engaging in formal deductive proof. At level 1, students recognise shapes as visual wholes. Individuals may attend to at least one feature of the shape but are not able to reliably identify all shapes in the family. At level 2, students begin to describe shapes by necessary properties using strictly informal language based on visual rather than conceptual knowledge. When formal language is used, it is to describe what they 'see' rather than what they have inferred. At level 3, students begin to look beyond the physical images. They describe and reason all the properties generally associated with a shape, including diagonal properties, angle and side features and parallelism. At level 4, reasoning is based on empirical evidence. Individuals progress from inferring that if a shape has one property, it has additional properties, to giving necessary and sufficient definition for a particular shape and, finally, recognise the hierarchy of shapes. Formal deductive reasoning takes place when students can construct arguments based on the properties of shapes.

These five reasoning abilities are seen as interconnected and develop progressively with various degrees of emphasis and importance depending on the task demand. They also apply to other areas of geometry such as three-dimensions (3D) and transformations of shapes. Proficiency in one domain is supported by a good ability developed earlier. Hence, the word 'progression', as opposed to 'trajectory', implying a single pathway, is used to reflect the nature of learning as moving within and across domains. Underpinning these abilities is the degree of connectedness among visual representations, visualisations and mathematical discourse. Visual representations possess both figural and conceptual characters (Fischbein, 1993). Figural characters can be external, embodied on paper or with other materials, or iconical, centred on visual images. Conceptual characters are the '*concept image*', the collective mental pictures and the corresponding properties and processes that are associated with the concept (Vinner, 1991). Such images are schematic, bound by their '*formal concept definitions'*, a form of words used to specify that concept (Tall & Vinner, 1981, p. 152) and developed through the process of visualising. Visualisation is a cognitive process in which objects are interpreted within the person's existing network of beliefs, experiences, and understandings (Phillips, Norris, & Macnab, 2010). Visualisation is needed across all levels of reasoning situations.

Individuals develop their own personal concept images and concept definitions through experience. Initially, geometric representations are understood purely from visual recognition. As an individual's knowledge deepens, so does their ability to look beyond the physical images, infer and deduce the geometric relationships the images present. Language used to describe shapes also moves from informal to formal, where eventually the student is able to provide a necessary and sufficient definition for a particular shape. Difficulties arise when there is a disjuncture between personal geometric knowledge (concept image and definitions) derived from experience and formal geometric knowledge deriving from axioms, definitions, theorems and proofs. In geometry, visual representations in the forms of points, lines, angles and shapes are used to take an abstract concept and make it concrete. Terms such as square, triangles and circles are condensations of definitions (Duval, 2014), not necessary when used in a designative or descriptive way but crucial when used to infer or justify a particular geometric argument.

Engaging in Mathematical Discourse

Since thinking is a form of communication and learning mathematics is about changing a discourse (Sfard, 2008), the way a student perceives and talks about geometric visual representations reveals their thought processes and in turn shapes their thinking. Mathematical discourse exhibits four interrelated characteristics:

- *Keywords* are used in distinctly mathematical ways to describe and define a particular shape. They reveal how a student sees and interprets that shape.
- *Visual mediators* are means with which participants in discourse identify the object of their talk and coordinate their communication.
- *Narratives* are a set of spoken or written utterances used in mathematical discourse that are subjected to endorsement or rejection, with the help of discourse-specific substantiation procedures.
- *Routines* are well-defined repetitive patterns in which mathematical tasks are being performed.

This framework allows any disjuncture between students' discourse and mathematical narratives that are 'taken-as-shared' within the mathematics community to be identified. In particular, the way in which *keywords* and *visual mediators* are used to construct meaning reflects the degree of disjuncture between a student's personal and formal concept definition, which in turn assists in the refinement of the learning progression. Since this is a self-reporting survey, the studying of classroom routines was not possible. Nevertheless, students' responses may provide a window into how mathematical tasks were performed in these classrooms.

Method

Teachers from one trial school (Year 8) and 37 (Years 7-9) project schools were given the task of asking their students to provide as much details as possible their knowledge of a square. The trial school was from an inner suburb of a capital city. Class A (quoted as CA) was given a regular curriculum and class B (quoted as CB) was on an accelerated curriculum. The school's mathematics coordinator mentioned that both classes have studied geometry prior to the commencement of this study although no details on the type of content students received were given. Three project schools (quoted as PM) provided the data. They were situated in low socio-economic areas from across Australia. No information was given on the type of geometry learning experience they had received. Data from a total of 214 students were collected and analysed. The exact number of students for each year level for the project schools is unknown as some students did not indicate their year level.

Initial analysis involved classifying aspects of the responses according to keywords. Further analysis considered the construction and substantiation of narratives, visual mediators used and possible routines in the discourse. Square is a special case in two-dimensional (2D) shapes. It is a regular polygon and links to other geometric concepts such as rectangle, rhombus, parallelogram and symmetry. Surveying the students' knowledge about a square allows the 'interface' between students' mathematical discourse and 'real-life' talk (Sfard, 2008, p.226) be analysed.

Findings

Keywords

A plane figure with four equal straight sides and right angles is the endorsed narratives for defining a square. Initial analysis focuses on the *keywords* students used to describe the lines forming the boundary of a square. As indicated in Table 1, class B students out performed the other two groups in using words such as 2D, quadrilateral and parallel sides to describe a square. Although more students from project schools stated that a square has 4 sides, this definition is insufficient. The idea of equal in length is important, and was mentioned by 30.8% class A, 50% class B and 52.4% project school students. The other necessary component of the definition for a square is right angle. As a multifaceted concept, angle may be defined as: a geometric shape, the union of two rays with a common end point, a movement, a rotation, a measure, and the amount of turns (Henderson & Taimina, 2005).

Table 1

Keywords used by the students to describe a square

Keywords	% of responses			Keywords	% of responses		
	CA	CB	Project		CA	CB	Project
2D	26.9	41.7	15.2	3D	7.7	12.5	1.2
4 sides	61.9	58.3	80.5	4 corners	50	20.8	43.9
lines	26.9	0	4.3	4 angles	7.7	41.7	12.2
4 edges	3.8	25	6.1	right angles	23.1	58.3	15.9
straight	23.1	12.5	4.9	90°	26.9	25	24.6
equal sides	19.2	37.5	36	points	7.7	0	0.6
even	11.5	8.3	9.8	vertex	3.8	25	1.2
same length	3.8	25	7.3	quadrilateral	11.5	50	4.9
1 face	11.5	16.7	4.3	Flat	0	0	3
Parallel sides	19.2	33.3	4.3	area and perimeter	0	16.7	6

Since angle derived from the Greek word *gonia*, to mean 'corners or knees', the word corner is acceptable for younger children whereas a formal and precise term is preferred for secondary years. The data showed that more students from the trial school mentioned corners and the class B students out performed others in specifying that a square has to have right angles. Moreover, two project school students compared with 3 class A and 10 class B students mentioned that the sum interior angles for a square is 360º. Collectively, 15.4% class A, 50% class B and 12.5% project school stated the equivalent of 4 equal sides with right angles. Although only one student from each group gave a full definition of 4 straight equal sides with right angles, analysis of *keywords* used showed that class B students have better understanding of the concept of a square.

Constructing and substantiating narratives

In geometry, narratives are constructed through knowledge of axioms and theorem and substantiated by deduction. There is evidence that visualisation played a large part in students' ability to visualise, describe, analyse, infer and deduce geometric relationships. Initially, direct identification or what Sfard (2008, p.228) termed as object-level utterance was based on a physical visual cue, as Year 7 PM4 claimed, 'everyone knows what a square is,' without even a need to provide a diagram. Around 7.3% of project school students made reference to their life experience such as tiles, Spongebob, Minecraft, napkin, used for building, and A5 paper is square (*sic*); 19% of class A students stated that it is a 6 letter word that begins with ‘S’ and two students mentioned that it is worth 15 points in Scrabble. Three class B students pointed out that squares are used in designs, puzzles and pixels.

The relationship between quadrilaterals is a difficult concept for many students and teachers alike. The data showed that students’ understanding varies from seeing a square being rotated to stand on a corner as a diamond (one class A, and three class B and project school respectively), to using the term rhombus (one project school and three class B students). Two project school students drew a parallelogram but did not provide an explanation, whereas four class B students stated that a square is a parallelogram. The term was not mentioned in any of the class A responses.

With regards to rectangle, 13 project school students saw a square as distinctly different from a rectangle:

> PM5: Can be any size or shape as long as vertical lines aren’t too long because then it would look like a rectangle.

Three students from class A and one from class B made similar statement. Only one project school student saw square as being ‘similar to rectangle’ whereas 45% class B students stated that a square is a type of rectangle. The limitation of surveying through self-reporting means that in-depth analysis of this knowledge can further be sought through interviews.

While students could construct narratives about square, project school students were found to make claims that were unsubstantiated and unendorsed mathematically, such as ‘it has 4 sides and it’s a square’ (Year 7, PM20). Some students’ image of square appeared stereotypical and static as the word ‘vertical’ and ‘horizontal’ were found in two class A and four project schools’ narratives, with project school students further having asserted that ‘it cannot move’. Limited transformation language was used as only two class A and four project school students mentioned symmetry. Three project school students also talked about reflection and one CB student mentioned tessellation.

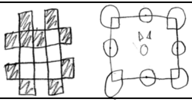
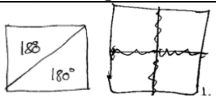
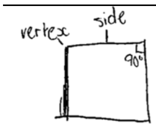
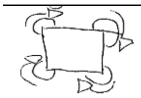
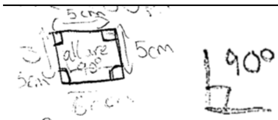
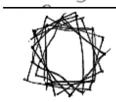
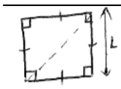
The use of visual mediators

In geometry, diagrams are schematic. They represent the broader, topological features of a geometrical object and form part of the logic of an argument (Netz & Noel, 2007). Engaging in geometrical conversation necessitates fluent use of diagrams to communicate ‘taken-as-shared’ mathematical discourse. The data showed that 34.6% class A, 50% class B and 30% of project school students used diagrams as mediators. However, many of the diagrams did not serve a clear purpose in their explanations, as nearly half of class A (44%) and 51% of project school drew a square with no embellishment. Conversely, only one class B student drew a simple square.

Engaging in geometric discourse requires individuals to ‘see’ the inherent geometric relationships in diagrams irrespective of orientation and communicate this knowledge using discourse-specific procedures. Students’ use of diagrams range from ‘real-life’ examples (seeing squares in hashtag), labelling or circling the positions of angles colloquially, to mathematical signifiers (Table 2). A cuboid with arms and a house with high square window or roof were among some of the diagrams drawn by project schools. No such diagrams were found in the trial school.

Table 2

Samples of diagrams students drew to explain the concept of square

Student’s drawing	Students’ descriptions
	PM 61: Well a square is a shape that contains four sids, and 4 corners. A square also hase 4 right angles, 4 reflex angles, 8 strate angle and 8 revulooshion. This shape is a hastag
	PM59: Square has 2 tringle in it the square equil 360°. You can half a square in diffrent ways. You can cut square in to rectangle. If you you put to 2 tringle together it will make square.
	PM75: A square is a 4 sided shape otherwise known as a quadrilateral. All 4 sides are the same length. It has 4 vertexes (corners). Each vertex equals 90°
	CA06: All the same.
	PM44: A square is a 4 evan sided shape and corners and 90° angles a square looks like this
	CA18: 90° of different rotations
	CB45: Consists of at least 2 triangles

Mathematical signifiers are words or symbols that function as nouns in utterances of discourse participants. As seen in Table 3, the type of mathematical signifiers used by class A were restricted to right angles and dissections of squares. A diagram is considered a dissection of a square when it was supported with a statement such as ‘it has 4 small square’ or ‘it can be divided into triangles’. The diagrams in class A showed dissection of square into four parts or eight triangles. Conversely, not only were class B students better at conveying the concept of right angles and equal length, their drawings were near identical. That is, students who drew a dissection of square did it with one diagonal line. Although both classes have studied some geometry, the results suggest that their learning experience were quite different. It could also be that students in class B were better at retaining knowledge learned than students in class A.

Table 3

The type of mathematical signifiers used by the students

Mathematical signifiers	Class A	Class B	Project Schools
Right angles	33%	92%	20%
Equal length	0	83%	14%
Lines to signify symmetry	0	0	3.6%
tessellation	0	0.08%	0.02%
transformation	11%	0	0
Dissection of square	33%	33%	0.06%
Cube	0	0	20%
Net	0	1	0

* Calculated based on the number of students who use diagrams as mediator per cohort.

Of the three groups of students, project school students produced the most diverse range of diagrams, including drawing squares to show tessellations, as well as drawing and specifying that a square has 4 lines of symmetry. Around 20% of project school students also drew a cube. Only one class B student among the cohort attempted to explain parallelism in his/her drawing despite of the fact that the concept was mentioned by three other class B and two project school students.

Is square a cube?

Students' confusion of square, a 2D concept, with 3D ideas were evident as words such as edges, vertex and face were used in their descriptions (see Table 1). For example,

> PM43: a square has 4 sides in 2D, 6 sides in 3D. Squares have 8 corners in 3D. Square have 12 edges.
>
> PM12: a 2D square has 4 sides and a 3D square has 6 sides
>
> CB56: 3D version is a cube.
>
> CA: Square is a 2D version of a cube.

Although no students in class A or B drew a cube, two class A and four class B students made reference to the idea. Of these six students, one class A student maintained that it is not a cube while the remaining cohort said that 'its 3D version is a cube'. Conversely, 20% of project schools who provided a diagram drew a cube in reference to 3D concept. Their responses range from: it's a cube (1.2%), can be 3D (4.8%), it's a 3D version of a cube (6%), it's not 3D but you can make it (1.2% responses with no substantiation), and it's a surface of a cube (1.2%).

> PM4: Goes into a cube if you get another square and put together

The students linked square to 3D in three ways. First, direct identification where students drew a cube and called it a 3D square, or square cube. Second, analytical identification where students recognised that a square is 2D but becomes a cube in 3D. Third, students made clear distinction and talk about the relationships between squares and cubes with the six faces of a cube being squares. Three students explained that squares are used to construct a cube, none drew nets to substantiate the claim. Four class B and two project school students made reference to square based pyramid. However, only one class B student drew a net and a solid to support the statement.

Discussion and Conclusion

Sfard's discursive framework enabled the 'how' and 'what' of learning to be studied: What has been learned? How was it learned? What were the outcomes? The results provided some answers to these questions. First, the data showed that not much geometry was taught across Australian schools, especially in relation to concept definitions, classes of shapes, transformation and 3D space. As such, only one student in each of the group was able to provide a necessary and sufficient definition. Students' discourse revealed the way they saw and interpreted square. The lack of exposure to geometric shapes in different situations, and the emphasis on visual and concept definitions hinder students' ability to 'see' beyond the obvious physical appearance. Hence, nearly all the diagrams drawn were stereotypical (as in □ instead of ◇). Mathematical signifiers were rarely used. Neglecting the teaching of geometry meant that there were few opportunities for students to develop inferential and deductive reasoning ability in school, which impacts on their ability to engage in mathematical discourse. Accordingly, even classroom B students who understood that a square is a rectangle did not think it was necessary to justify this claim.

In light of the range of responses across the cohort, it is clear that geometric knowledge is more experiential then developmental. This is shown in the identical drawings done by class B. Conversely, despite being placed in a regular curriculum, CA18 was able to use diagram to show the concept of transformation, a concept that was overlooked by students in class B. By mapping the range of responses using the learning progression, the results indicated that the majority of the students were reasoning at descriptive and analytical levels. While this mirrors studies conducted using van Hiele's levels, the results reported here indicated that the relationship between visual representation, visualisation and mathematical discourse is reflexive. A lack of mathematics discourse hinders communication, hence mathematical thinking, as reflected in the number of students who drew a square without explanation. In view of the fact that these abilities are interconnected and develop progressively as students' discourse becomes increasingly mediated, the next stage of the research is currently underway, where assessment and learning tasks are designed to encourage the cultivation of inferential reasoning. Further data collected will assist in the refinement of the learning progression.

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