Developing Conceptual Understanding of Fractions with Year Five and Six Students

Judith Mills
The University of Waikato
judith@waikato.ac.nz

This paper presents findings from classroom observations of one teacher (Beth). It focusses on the development of conceptual understanding of fractions with her students, articulated in Kieren's sub-constructs (Kieren, 1980,1988), and Hansen's progressions (Hansen, 2005). The study covers three lessons within a six week unit. Findings from this study suggest there is a need for teachers to understand and teach a wider range of fraction constructs than appears to occur currently; how assessment data available might be used more effectively to plan quality lessons; and highlights the importance of a varied range of models to connect conceptual and procedural understanding of fractions.

Theoretical Background

Professional knowledge required for teaching is complex and multi-layered, covering many different aspects of knowledge required by teachers in relation to the students they teach, the subject they teach, and the manner in which they teach. While it requires knowledge of what is taught and how it is taught, it also requires knowledge of how students think and what they understand, before they learn the subject matter as well as during the learning process. Therefore teachers must have knowledge about how mathematics is learned, how topics should be sequenced for learning, where conceptual blockages occur, and where misunderstandings are likely (Barton, 2009).

Prior to the 1950s school mathematics focussed on computation and it was with introduction of computational tools, and the associated requirement of higher-order thinking skills, that the need for people to be able to transfer their mathematics understandings to everyday life increased (Perso, 2006). Perso advocated that mathematics must be taught in school in a manner which allows students to make connections to life beyond the classroom, while Hiebert and Carpenter (1992) suggested that something is understood if it can be related, or connected, to other things that are known. As a consequence, in today's mathematics classrooms concepts are usually taught first and foremost. Procedures are also learned, but alongside a conceptual understanding. Once conceptual understanding is gained, then a person can reconstruct a procedure that may have been forgotten. Students can represent the mathematical knowledge they learn in various ways using one, or a combination, of five elements including spoken language, written language, manipulatives, pictures, and real-world situations (Lesh, Landau, & Hamilton, 1983). Concrete manipulatives are often utilised to support conceptual understanding and teachers must have a clear understanding of how they can be utilised in order to assist in the children's learning process (Swan & Marshall, 2010). If the teacher and students do not explain and understand their use of the tools (manipulatives), then teachers are in danger of replacing verbal rules and procedures, with rules and procedure for using tools (Yackel, 2001). Swan and Marshall defined manipulatives as "an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered" (p. 14).

Fractions is a topic which many teachers find difficult to understand and teach (Chick, 2010; Post, Cramer, Behr, Lesh & Harel, 1993) and consequently many students struggle

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with learning basic fraction concepts at the primary school level (Anthony & Walshaw, 2007; Davis, Hunting, & Pearn, 1993; Lamon, 2007). There appears to be a difference between students' conceptual and procedural understanding of fractions and being able to connect intuitive knowledge and familiar contexts, with symbols and formal classroom instruction (Hasemann, 1981). The models students experience either visually or mentally, relate to students' written solutions to problems, and the ability to interpret and represent visual static models of fractions, may be a precursor to understanding fraction concepts (Anderson-Pence, Moyer-Packenham, Westenskow, Shumway, & Jordan, 2014). Developing intuitive models to assist teachers and students to understand quotitive division with whole numbers, is required to assist teachers in responding to word problems involving fractions (Roche & Clarke, 2009).

Much of the confusion in teaching and learning fractions appears to arise from the many different interpretations (constructs) and representations (models) available. Kieren (1976) originally introduced the idea of seven sub-constructs of rational number and later revised these to four sub-constructs based on part-whole conceptions, which included measures, multiplicative operators, quotients, and ratios (Kieren, 1980; 1988). While Kieren emphasised that the part-whole construct was the foundation to students' learning about rational number, he stressed the need for students to integrate the sub-constructs. He suggested that the sub-constructs should not be viewed in isolation or independent of each other, but rather that they combined to create a generalisation of rational number. However, it has also been suggested that initially the key interpretations of fractions need to be taught in the order of conceptual challenge, for understanding to occur (Hansen, 2005). The order of conceptual challenge that Hansen suggests for understanding to be embedded in practice is: part of a whole (an object is split into two or more equal parts); part of a set of objects (what part of the whole set of objects has a particular characteristic); numbers on a number line (numbers represented between whole numbers); operator (the result of division); and finally ratio (comparing the relative size of two objects, or sets of objects).

The above types of fractional understanding presented by Hansen (2005) are based on progressively increasing levels of complexity, and are required in order to carry out operations on, and with fractions. Although these constructs may be considered separately they are unified by three 'big ideas': identification of the unit; partitioning; and the notion of quantity (Carpenter, Fennema, & Franke, 1996). Young students' fraction knowledge will often begin with the teaching of the part-whole concept where, for example, it is understood that $\frac{3}{4}$ is three parts taken from a whole that has been partitioned into four equal parts. However, there is a need for school experiences to support student conceptions beyond the part-whole construct.

The main purpose of this study was to examine how Beth developed conceptual understanding of fractions among her students throughout a unit of work. It looked at the way in which recognised sub-constructs of fractions (Kieren, 1980, 1988) were integrated, through learning progressions (Hansen, 2005), and what fraction learning and understanding occurred as a result of this. The key question of the study was: How do teachers develop conceptual understanding of fractions with their students, which will allow them to transfer that knowledge into everyday life?

Method

Participants: Beth taught 21 students in Years 5 and 6 (comparable to Grades 4 and 5 in most other countries - approximate ages 9 to 11 years) at an urban New Zealand primary school. There were six Year 5 and 6 classes at the school and they cross-grouped by ability for mathematics. Grouping was decided according to prior assessment data, which included results of Progress and Achievement (PAT) tests (New Zealand Council for Educational Research, 2006) and Numeracy Project assessment tool (Ministry of Education, 2008). Beth's class was the fourth class in ranking out of the six.

Procedure: Data were gathered from initial and end of unit assessments, along with in-class observations. Classroom observations were the key part of data collection, which focused on three lessons within a six week unit on fractions. The first lesson observed was during the first week of the unit, while the other two lessons came intermittently throughout. In order to validate the observations of the lessons, field notes were written, photos taken, and lessons both audio-taped and video-recorded. This allowed the researcher to return to the lessons and cross-check details at a later date.

Pre-unit and post-unit assessment tasks designed by the researcher were based on key aspects of knowledge students at Years 5 and Years 6 are expected to implement according to Level three of the New Zealand Curriculum (Ministry of Education, 2007) and the National Mathematics Standards (Ministry of Education, 2009).

Results

Assessment Tasks: Six assessment tasks to ascertain current fraction knowledge and understanding were presented to the students prior to the commencement of the unit. Similar tasks were repeated at the conclusion of the unit (Table 1) and became part of the evidence of progression of learning. With conceptual understanding of fractions the focus, the students were asked to solve each problem, explain how they worked it out, and where possible draw a diagram to show their thinking.

Table 1
Number and Percentage (in brackets) of students with correct responses on the assessment tasks

	Task 1 Addition of unit fractions	Task 2 Addition of fractions with compatible denominators	Task 3 Word problem comparing fractions	Task 4 Part to whole – unit fractions	Tasks 5 Whole to part – fraction of a set	Task 6 Order fractions smallest to largest
Pre-unit assessment n = 21	2 (10)	0	2 (10)	10 (48)	4 (19)	1 (5)
Post-unit assessment n = 20	13 (65)	0	9 (45)	13 (65)	9 (45)	2 (10)

On both assessments, Task 4 had the highest number of correct responses. This task related to 'part-to-whole' thinking. The question on the initial assessment was: *If $\frac{1}{4}$ of my circle has 3 smiley faces, how many are there on the whole circle? How do you know?* The final assessment had 5 smiley faces on each quarter. Most students drew quarters onto the circle and placed 3 faces on each quarter, then either added 3 + 3 + 3 + 3 = 12, or in some instances used multiplication: 4 x 3 = 12. The students struggled to accurately add fractions with compatible denominators (Task 2, $\frac{1}{10}+\frac{2}{5}$), with the most common error being the 'add across' error (adding numerators and denominators to get an answer of $\frac{3}{15}$). There was a noticeable increase in the number of students who solved this type of problem correctly at the end of the unit. Another significant increase in the number of correct responses was related to 'whole-to-part' thinking (Task 5), where on the final assessment the students were asked to find $\frac{1}{3}$ of 21. One student ordered the fractions on Task 6, ($\frac{1}{3}$, $\frac{6}{4}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{2}$) correctly at the start, and two at the end.

Lessons Observed: Beth began each lesson by sharing with the students the learning intention, and unpacked the meaning to ensure the students understood the purpose of the lesson. Manipulatives formed a fundamental part of the lesson. The initial lesson observed was based on addition of fractions with the same denominators, including fractions greater than one. Beth began the lesson by revising the meaning of the written fraction symbol and ensured the students understood representation of the numerator and denominator. The students then worked in pairs with each pair having three strips of paper of exactly the same size (representing wafer biscuits), to share out evenly between the two of them.

Beth: Do you want to tell us what you did?
Child: What we actually did was [um] we put them all together. We just cut them [all] in half and split them up and we had three each.
Beth: You cut them into halves?
Child: Yes, and got three each.
Beth: Three. How would we write that? How do you write that you have three halves?

The students thought for a while and the conversation continued:

Beth: You cut yours into halves and then you had one half one half one half? So, how many pieces did each person get? How many pieces have you got?
Child: Three.
Beth: Three and what are they [points to a whole] cut into?
Child: Two.

Eventually the students added the one-half, one-half and one-half, and after questioning by Beth they realised the three halves could be written as $\frac{3}{2}$. Discussion took place which focused on understanding that $\frac{3}{2}$ was greater than one whole unit. Two pairs of students then joined together to form a group of four, with three wafer biscuits (strips of paper) to share among the four people. The students soon realized that this time there were more people than wafers, so everyone would get less than one whole biscuit. Observation showed that there was some confusion over whether to cut the wafers into three or four pieces. Those who had cut the wafers into three pieces gave everyone two, which left one piece ($\frac{1}{3}$) to share among four people. They recognized that they needed to cut that piece into four equal portions, but had no idea what the name of that piece was. They knew each person had $\frac{2}{3}$ and 'a little piece'. The students who had cut each of the three wafers into quarters, shared all of the pieces out evenly, and recognized that they had three pieces, and each was one-quarter, therefore they had three-quarters.

Lessons 2 and 3 were based on finding fractions of a set. At the start of Lesson 2, Beth checked to see whether the students remembered the meaning of improper fractions and revisited the concept which had been explored in the earlier lesson. From here Beth moved onto finding fractions of a set. Each pair of students was given 20 multi-link cubes, which represented a 20 piece worm. They were to share their worm out equally among four birds and decide how many pieces of worm each bird would receive. Most of the students decided each bird would get five pieces of the worm. However, confusion arose when determining the name of the fraction of the whole worm they had and how many pieces that fraction represented. For example:

> Beth: This is a worm and it's made up of twenty cubes. And there are four birds that we want to share this worm with. How many pieces are they [each] going to get?
>
> Child: Five each.
>
> Beth: They get a quarter each. So can you show me what a quarter of your worm is? So (child's name), what is a quarter of 20?
>
> Child: Four.
>
> Beth: A quarter is how many pieces? A quarter of this twenty is?
>
> Child: Is five.

Similar difficulties arose when they tried to share a 15 piece worm equally among three birds, or a 12 piece worm was shared equally among three birds. When one group shared the 12 pieces of worm among three birds, they gave each bird three pieces and threw the rest away. The number three was the predominant number in their minds and they decided that the rest of the pieces were not required. Beth noticed this and reminded them that the 'whole' worm had to be shared out and they must decide what to do with the other pieces. Eventually they gave each bird another piece (to use them all up), but confusion was still evident as to what the four pieces represented. Did each bird have quarters or thirds?

During Lesson 3, the context for the problem examples was birthday cakes with 'jelly-beans' on the top. Beth began with a cake cut into fifths and 25 jelly-beans to spread equally on the five portions, therefore how many jelly-beans would there be on each piece? Five jelly-beans representing one-fifth meant the number 'five' was dominant in this problem. From there the students solved, 21 jelly-beans on a cake cut into thirds. Beth then moved on from unit fractions and asked the students how many jelly-beans would be eaten if someone ate three-quarters of a cake, which had 12 jelly-beans spread evenly on each quarter of the cake.

Towards the end of the lesson Beth changed from the 'whole-to-part' concept, to 'part-to-whole'. She began with, "*If we have six candles on one-third of a cake, how many candles are on the whole cake?*" Some of the students solved the problem using addition (6 + 6 + 6 = 18), while others solved it using multiplication (3 x 6 = 18). Various other part-to-whole problems were solved involving fifths, quarters, and thirds. In order to extend some of the students, Beth finished with the lesson with the problem, "*Two thirds of the number I am thinking of is the number is eight. What is the number?*" Initially the students called out answers including: "four," "six," and "two". Beth gave the students a paper circle to represent the whole unit. She asked them divide the circle into thirds and spread eight beans evenly on the two-thirds. The conversation then went:

> Beth: You've got two thirds and you've got eight, so how many is there going to be on the whole cake?
>
> Child: Eight.
>
> Beth: But you haven't got any on this piece (pointed to the other third).
>
> Child: Oh, twelve.

Discussion and Conclusions

Assessment Tasks

The pre-unit assessment results showed that generally the students were below, and in many instances well below, their expected levels (Ministry of Education, 2009). With less than 50% of the students solving any of the initial Tasks correctly, this should have been an indication to Beth that all of the fundamental fraction concepts as articulated by Kieren (1980, 1988) required further exploration and understanding.

The post-unit assessment indicated that some progress had taken place throughout the six weeks. Beth had focussed much of her teaching on what Hansen (2005) identified as the two initial constructs to be taught – 'part of a whole', and 'part of a set' of objects, along with 'operators'. Connections had also been made to Kieren's (1980) two sub-constructs of 'multiplicative operators' and 'quotients'. This resulted in a large increase in the number of students solving the problems on Tasks 1, 4, and 5 correctly.

Given that no student could add compatible fractions accurately, introducing equivalent fractions could possibly have been planned for within the unit. Thirds and sixths, or halves, quarters, and eighths, might have been unpacked alongside each other when sharing items out evenly. Comparing the smaller-sized fraction piece to its compatible fraction (sixths to thirds, and eighths to quarters), along with discussion based on the positioning of fractions on the number line, would have consolidated understanding of the fraction number and assisted students with Task 6 (ordering fractions from smallest to largest). The inter-relationship of these constructs was emphasised when Kieren (1980) stressed the importance of not continually teaching them in isolation but integrating them in order to fully understand rational number.

Observed Lessons

The three lessons observed were based on the big ideas of fractions as expressed by Carpenter et al. (1996) as being 'identification of the unit', 'partitioning' and the 'notion of quantity'. These ideas began with unpacking the writing of unit-fractions, non-unit fractions, improper fractions, and mixed numerals consolidated for the students the meaning of the written fractional number. While some of the students took the next step of converting the $\frac{3}{2}$, to $1\frac{1}{2}$, Beth did not insist that they do this as she wanted the students to realise that fractional numbers can be written as improper fractions.

When the students shared 3 whole units among 4 people difficulty arose (for some) in recognising which number represented the divisor and which number the dividend, therefore the number of groups that were to be formed, and how many should be in each group: Should it be 3 groups or 4 groups, with 3 or 4 in each? The combination of the use of manipulatives and real-life problem scenarios assisted the students in solving this dilemma. Concrete manipulatives play an important role in developing conceptual understanding (Swan & Marshall, 2010) and as Perso (2006) advocated, modelling the real-life scenario provided meaning to the given problem. When given the opportunity to cut the 'wholes' into pieces and share them out, the students recognised whether they had formed enough 'piles' for the number of people. Having the appropriate types of manipulative for the given problems became apparent (strips of paper, paper circles), as rigid commercially made fractional pieces (which were in a box on the floor next to Beth) would not have given the students the same opportunity and flexibility to solve for themselves how to share the pieces 'of wafers' out among the four people. As Yackel (2001) stressed, having the correct manipulative to explain the representation is crucially important or students will use the manipulative procedurally, rather than with conceptual understanding. When Beth gave the students a 'part-to-whole' problem to solve, which was not presented in a real-life scenario or manipulatives made available, the students struggled to make sense of the problem. Given paper circles and 'beans' to manipulate, the students solved the problem step-by-step and unpacked a correct solution to the problem.

Once the students had shared out the original whole units, many of them were unable to name the 'fractional representation' that had been formed. For example, the students who ended up with $\frac{2}{3}$ and a 'little bit' each, had no idea that the little bit represented $\frac{1}{4}$ of $\frac{1}{3}$ or $\frac{1}{12}$. Confusion and difficulty arose in the re-unitising of the pieces and the relationship between the number that resulted, and the number of wholes they started with. For example, when three whole (wafers) were shared among four people, each person received three one-quarter sized pieces, which were also $\frac{3}{4}$ of the original whole (three wafers).

The students were able to solve the 'part-to-whole' problems with more ease than the 'whole-to-part' problems. Part-to-whole problems involve multiplication, while solving whole-to-part problems involved division. Past research has shown that students' understanding of division is more problematic than that of multiplication (Roche & Clarke, 2009), and without a strong understanding of division with whole numbers, students will struggle to understand Kieren's (1980) construct of 'quotients' and the relationship with other constructs. Division of fractions often relates to the understanding of quotitive division, a concept which Roche and Clarke found many teachers and students struggled with. This was exemplified here when the students lost sight of the 'groups of' idea associated with whole-to-part division.

The need for students to transfer their mathematical understanding into real-life situations is becoming increasingly paramount, as they prepare for a technologically-based work force in later years (Perso, 2006). Students in this study struggled to understand fraction concepts when they were presented in isolation, and the combination of real-life scenarios and manipulatives assisted in making their mathematics meaningful. This study showed that the inter-relationship between Kieren's sub-constructs of fractions (Kieren, 1980, 1988) is integral to effective pedagogical practice and a key to understanding rational numbers. However, the findings of this study were limited by the teaching of three lessons by one teacher. Of further interest would be the comparison of these to a wider cohort of teachers over a greater number of lessons.

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