Investigating the Interconnections between Cognitive, Affective and Pedagogical Issues in the Learning of Group Theory

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Undergraduate mathematics students consider Group Theory as a challenging topic. This study aims to investigate the interrelation of cognitive, affective and pedagogical issues of students' first encounter with this module. The results suggest that there is interdependence between cognitive difficulties, affective reactions, involving a wide spectrum of emotions, and pedagogical activities, particularly the teaching and learning processes, in relation to coping strategies. This "triangular" interdependence of the three aspects is described as the "trilateral interlock of learning".

Background

Group Theory is considered by the majority of undergraduate mathematics students as one of the most demanding modules in their curriculum, in which they face both cognitive and metacognitive challenges (Ioannou, 2012). Often, after their first encounter with Group Theory, students tend to avoid third-year or further courses in this area of mathematics. Nardi (2000) attributes student difficulty with Group Theory to its multi-level abstraction and the less-than-obvious, to students, raison d'être of concepts such as cosets, quotient groups etc. Dubinsky et al (1994, p. 268) have concluded that students after their first encounter with it, they avoid any further study, since it is "the first course in which students must go beyond 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes."

In addition, Group Theory requires "deeper levels of insight and sophistication" (Barbeau, 1995, p. 139). This challenge is also due to the fact that instructors, more often than not, do not give adequate time to students to reflect on the new material (Clark et al, 1997). Weber (2001, 2008) and Ioannou and Nardi (2010) suggest that cognitive difficulties are related to metacognitive issues concerning students' coping strategies in the learning process, for example proof production and consequently affective issues. Additionally, the students' introduction to the novel ideas of groups takes place in the unfamiliar academic context of large-scale lectures. This unfamiliarity is likely to exacerbate their difficulty with the topic (Mason, 2002, p. 52). Also, as it is often suggested by research (e.g. Millet, 2001), lecturing to large student audiences has an arguable effect on student engagement.

Research on how mathematics undergraduates cope with the complexity of university studies suggests that in proof production, an essential part of their studies in mathematics and "the only means of assessing students' performance" (Weber, 2001, p. 101), students face difficulties of two categories: firstly, "they do not have an accurate conception of what constitutes a mathematical proof", and in addition they "may lack an understanding of a theorem or a concept and systematically misapply." (Weber, 2001, p. 102)

This study examines the interconnection between pedagogical activities (teaching and learning) cognitive difficulties and affective reactions that undergraduate mathematics students experience in their first encounter with Group Theory. This

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interconnection is described as the *trilateral interlock of learning*. For this purpose, I closely examine a typical case of a second year undergraduate mathematics student named Calaf (pseudo name), using the Commognitive Theoretical Framework (Sfard, 2008) for analysing cognitive difficulties, and Goldin's Framework (2000) to investigate the emerging emotional reactions.

Commognitive Theoretical Framework

Commognitive Theoretical Framework (CTF) involves a number of different constructs such as metaphor, thinking, communication, and commognition, as a result of the link between interpersonal communication and cognitive processes, with commognition's five properties *reasoning*, *abstracting*, *objectifying*, *subjectifying* and *consciousness* (Sfard, 2008).

In mathematical discourse, unlike other scientific discourses, objects are discursive constructs and form part of the discourse. Mathematics is an *autopoietic system* of discourse, namely "a system that contains the objects of talk along with the talk itself and that grows incessantly 'from inside' when new objects are added one after another" (Sfard, 2008, p. 129). CTF defines discursive characteristics of mathematics as the *word use*, *visual mediators*, *narratives*, and *routines* with their associated *metarules*, namely the *how* and the *when* of the routine. In addition, it involves the various objects of mathematical discourse such as the *signifiers*, *realisation trees*, *realisations*, *primary objects* and *discursive objects*. It also involves the constructs of *object-level rules* and *metadiscursive level* (or *metalevel*) *rules*, along with their characteristics *variability*, *tacitness*, *normativeness*, *flexibility* and *contingency*.

Mathematical discourse involves certain objects of different categories and characteristics. *Primary object* (p-object) is defined as "any perceptually accessible entity existing independently of human discourses, and this includes the things we can see and touch (material objects, pictures) as well as those that can only be heard (sounds)" (Sfard, 2008, p169). *Simple discursive objects* (simple d-objects) "arise in the process of proper naming (baptizing): assigning a noun or other noun-like symbolic artefact to a specific primary object. In this process, a pair- noun or pronoun, specific primary object- is created. The first element of the pair, the signifier, can now be used in communication about the other object in the pair, which counts as the signifier's only realization". *Compound discursive objects* (d-objects) arise by "according a noun or pronoun to extant objects, either discursive or primary" (Sfard, 2008, p166).

The (discursive) object signified by S in a given discourse is defined as "the realization tree of S within this discourse" (Sfard, 2008, p. 166). The realization tree is a "hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations and so forth" (Sfard, 2008, p300). Realisation trees and consequently mathematical objects are personal constructs, although they emerge from public discourses that support certain types of such trees. Additionally, realisation trees offer valuable information regarding the given individual's discourse (Sfard, 2008).

The epistemological tenet of CTF described in the last sentence is cardinal in its development as theoretical framework. Due to this tenet, Sfard (2008) describes two distinct categories of learning, namely the object-level and the metalevel discourse learning. Moreover, according to Sfard (2008, p. 253), "object-level learning […] expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse". In addition, "metalevel learning, which involves changes in metarules of the discourse […] is usually related to exogenous change in discourse. This change means that some familiar tasks, such as, say, defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses" (Sfard, 2008, p. 254).

Goldin's Theoretical Framework

For the analysis of students' emotional reactions, I am using the framework of Goldin (2000), which describes affect in mathematics in terms of four elements: beliefs and belief structures; attitudes; emotional states; and, values, ethics and morals. Particularly significant to this study is his notion of *local affect*, "the rapidly changing states of feeling that occur during problem solving – emotional states, with their nuances" (p. 210). Goldin describes eight such emotional states and several possible ways in which these affective states may lead to certain problem-solving strategies. Data partly refers to problem solving – as much of the student experience in Group Theory revolves around their engagement with the problem sheets. However, data also involves direct evidence of students' affective responses to Group Theory (for example in the lectures and the seminars) as well as accounts of these affective responses (for example in the interviews).

According to Goldin (2000) at the first stage of problem solving the student is likely to experience feelings such as *curiosity*, *puzzlement* or *bewilderment*. Following this there are two possible affective pathways: *favourable* (i.e. emotions of encouragement, pleasure, elation and satisfaction) and *unfavourable* (i.e. emotions of frustration, anxiety and fear/despair). *Affective pathway* is defined as "a link between one's affective and cognitive representation systems (i.e. a consistent cognitive response to an emotional state)" (Weber, 2008, p. 82). These pathways in local affect lead to *global affective structures* such as specific representational schemata, general self-concept structures as well as (particularly the second pathway) self – mathematics – science – technology resentment. Weber (2008) suggests that these affective pathways may be *self-strengthening* if their duration is long. A repeated emotional experience is possible to cause stable attitudes and beliefs that may be related to particular cognitive responses (Goldin, 2000). Moreover, as suggested by Weber, mathematical understanding is *organic*, since, when students feel that they have achieved some understanding in one mathematical topic and consequently they find it pleasurable, they want to extend their understanding with regard to this and other mathematical topics.

Methodology

This study is a ramification of a more comprehensive study, focusing on cognitive and pedagogical issues regarding undergraduate mathematics students' first encounter with Group Theory (Ioannou, 2012). Data collection took place in the Spring Semester of a recent academic year. The course was mandatory and attended by 78 students (10 weeks, 20 hourly lectures, 3 cycles of seminars, in Weeks 3, 6 and 10). The role of the seminars was mainly to support the students with their coursework. There were 4 seminar groups, with about 20 students in each. In the lectures, the lecturer was writing extensively on the chalkboard and was commenting orally alongside. In the seminars the students were expected to work on problem sheets, distributed to them earlier in the preceding weeks, and arrive having prepared questions. They had the opportunity to ask the seminar leaders and assistants anything they had encountered difficulty with and to receive help. The lecturer was also available during “office hours” for the same purpose.

Data consists of: Lecture observation notes; Lecture notes (notes of what the lecturer was writing on the blackboard); Audio-recordings of the 20 lectures and 24 seminar sessions; and interviews with 13 out of the 78 students who made themselves available on a voluntary basis, and with the 4 seminar leaders and assistants and the lecturer. There were three cycles of interviews, at the beginning, middle and end of the course. Additionally, data includes student coursework, which was handed in Week 12 marker (seminar assistant) comments on student coursework, and student examination scripts collected at the end of the academic year.

Lecture observation notes and lecture notes have been summarised and critically commented upon in an overview descriptive document, seminar audio-recordings have been summarised into lists of seminar vignettes, and student and staff interviews have been fully transcribed, and coded into a list of emerging themes and further analysed. Coursework and examination scripts of the 13 students, who have been interviewed, led to the identification of key themes such as the ones described below. Finally, all emerging ethical issues during the study have been addressed accordingly.

Data Analysis

In what follows, I examine the cognitive difficulties, the affective responses and the pedagogical activities as these emerge in the learning of Group Theory, focusing on a typical case of a second year mathematics student named Calaf. Calaf is an Eastern Asian student, educated in England from an early age. He has passed the course getting 50% in the coursework and 47.5% on the final examination.

First Component: Cognitive Difficulties

Scrutinising Calaf’s data from the coursework and final examination, there have emerged four instances of cognitive difficulties, typical to the majority of students. The first is related to the *misapplication of the test for a set to be a subgroup*. According to the theorem, if $(G,*)$ is a group and $H \subseteq G$, then H is a subgroup of G, if and only if: first $H \neq \emptyset$, second H is closed under $*$, i.e. $\forall \boldsymbol{h_1}, \boldsymbol{h_2} \in H, \boldsymbol{h_1 h_2} \in H$, and third H is closed under inverses i.e. $\forall h \in H, \boldsymbol{h^{-1}} \in H$. Calaf was successfully addressing the last two conditions, but he was persistently omitting to address non-emptiness, both in the coursework and the exam.

Overall, Calaf demonstrated correct and precise word use and use of visual mediators, composing a clear and logical narrative for the solution, apart from the first condition of the required routine. As the following excerpt suggests, he is not ignorant of the first condition of the routine, but he rather considers it immaterial to prove.

> The third question – you just use the um – I think the theorem? – to prove that. There’s – It’s not empty set – it’s – it’s not equal to the empty set. And then you prove that there’s – there’s x’s and inverse, or something, I don’t know, to prove that yeah. So you just use the theorem to prove it.

The second cognitive difficulty appears when Calaf has to *describe the group G of rotational symmetries of cube*, list them and show that there are 24 symmetries. Moreover, he has to show that the set of rotational symmetries of the cube, which send this pair of faces to itself, forms a subgroup of G of order 8. Again, he successfully used the group theoretic vocabulary as well as he visually represented the different symmetries of the cube. What his solution lacked was to apply the routine for a set to be a subgroup. This indicates that the related realisation tree is not yet endorsed and Calaf does not yet have the facility to apply it in different contexts.

The third cognitive difficulty occurred in the solution of an exercise asking for *proof of equivalence relations*, in the second problem sheet: Suppose X is a non-empty set and $G \leq Sym(X)$. Define a relation $\sim$ on X by: $x \sim y \Leftrightarrow$ there exists some $g \in G$, with $g(x) = y$. In this case, Calaf seems to misunderstand the given condition. As the following excerpt suggests, he seems to have difficulty to understand the narrative of the exercise and in his solutions there are instances where word use and visual-symbolic mediators are used incorrectly.

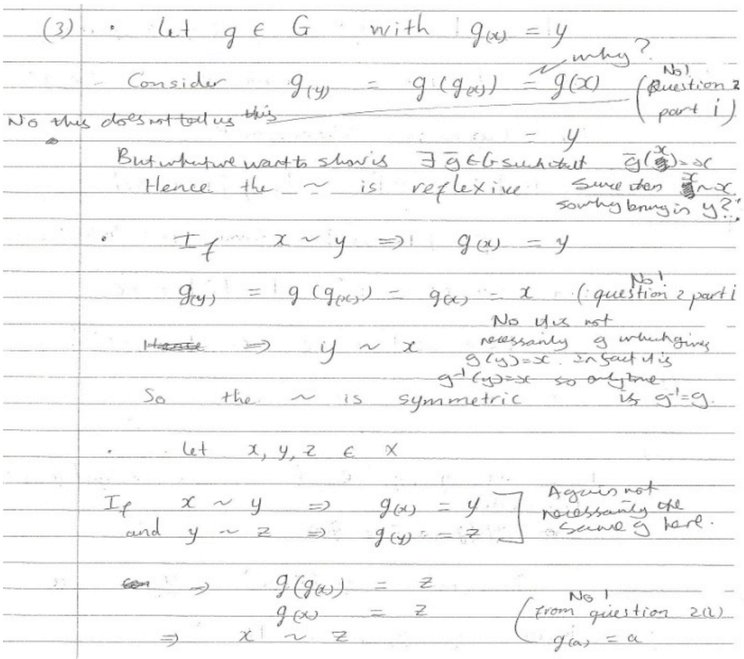
(3) • Let $g \in G$ with $g(x) = y$
Consider $g(y) = g(g(x)) = g(x)$ — why? — No! (Question 2 part i)
No this does not tell us this
$= y$
But what we want to show is $\exists\, \tilde{g} \in G$ such that $\tilde{g}(x) = x$
Hence the $\sim$ is reflexive — Sure does $\exists\, x \sim x$, so why bring in y?
• If $x \sim y \Rightarrow g(x) = y$
$g(y) = g(g(x)) = g(x) = x$ (question 2 part i — No!
No this not necessarily g which gives $g(y) = x$. In fact this $g^{-1}(y) = x$ so only time is $g^{-1} = g$.
~~Hence~~ $\Rightarrow y \sim x$
So the $\sim$ is symmetric
• Let $x, y, z \in X$
If $x \sim y \Rightarrow g(x) = y$
and $y \sim z \Rightarrow g(y) = z$ — Again not necessarily the same g here.
~~So~~ $\Rightarrow g(g(x)) = z$
$g(x) = z$ — No! (from question 2(i) $g(a) = a$
$\Rightarrow x \sim z$

Figure 1. Proving equivalence relations

As the above excerpt suggests, he seems to have adopted the correct routine i.e. to prove reflexivity, symmetry and transitivity; yet in the actual performance the application of the relevant metarules is incomplete. Probably this is due to incomplete endorsement of the group operation, including many aspects, namely, how inverses work, and how elements as such work (the last is confirmed by the fact that in the examination script, in a similar context, instead of writing $\forall a \in G$ he wrote $\forall a \in g$). In symmetric and transitive equivalence relations it is obvious that he uses incorrectly the conditions given by the exercise.

One last cognitive issue that emerged examining Calaf's data is related to the *proof that a relation between two groups is a homomorphism*. In two exercises he is asked to prove that the following two are homomorphisms, state in each case what is the kernel and the image of φ , and finally check whether the homomorphisms are in fact isomorphisms: first, $\mathbf{G}$ is any group, $\mathbf{h} \in \mathbf{G}$ and $\varphi\colon \mathbf{G} \to \mathbf{G}$ is given by $\varphi(\mathbf{g}) = \mathbf{hgh^{-1}}$, and second, $\varphi\colon (\mathbf{R}, +) \to (\mathbf{C}^{\times}, \cdot)$ given by $\varphi(\mathbf{x}) = \cos\mathbf{x} + \mathbf{i}\sin\mathbf{x}$. In both exercises, he successfully applies the routine for a group relation to be a homomorphism. He shows awareness of the definition of image and kernel, and correctly states them for the first one, but with no further explanation about the issue of isomorphism. What he was expected to write is the following: As $\varphi(\mathbf{g}) = \mathbf{e} \Leftrightarrow \mathbf{hgh^{-1}} = \mathbf{e} \Leftrightarrow \mathbf{g} = \mathbf{e}$ the kernel of g is the trivial subgroup $\{\mathbf{e}\}$. Moreover, as $\varphi(\mathbf{hgh^{-1}}) = \mathbf{g}.\boldsymbol{\varphi}$ is onto. Thus φ is an isomorphism. The absence of this bit from his solution shows either misunderstanding of the definition of the concept of isomorphism, or difficulty to apply the routine for a homomorphism to be an isomorphism.

Concerning the second exercise, he manages to prove that it is a homomorphism. In his solution occur several misunderstandings, for example he confuses the codomain with the image of the homomorphism, and he does not use the exponential version (i.e.$\varphi(\mathbf{x}) = \mathbf{e^{ix}}$) of the image, causing difficulty in the solution. Moreover, it occurs problematic word use, difficulty to use symbolic visual mediators (i.e. exponential expression) as a means for solving the exercise, as well as an overall incomplete application of metarules in the solution.

Second component: Affective reactions

In parallel to the apparent cognitive difficulties, there have emerged affective reactions, whose evolution appears to be interlinked. The first reaction of Calaf towards Group Theory involved emotions of curiosity and intention for engagement. Later on the affective pathway follows an unfavorable development with emotions of frustration and disappointment, which lead to partial disengagement. The following quotations give a holistic description of the evolution of his feelings, mostly in a problem-solving situation, but also towards Group Theory and Mathematics in general.

> You don't looking forward to the lecture any more, and you sort of turn up, when the lecturer talking in the lecture, so you just sort of listen, and you copy? But you sort of don't understand what going on. … I asked my friends, like three of the boys sitting around me, I said – do you know what it means, they said no, I don't, so they just copy it down as well, so I thought it's not only me, it's probably everyone else as well ... I spend the whole day doing work and then it's – didn't come out anything, you know what I mean, but not that – not the idea of the course ... And you spend the whole day, you come up with no results, and you thought – what's the point of doing any of this sort of thing, you know what I mean – you sort of feel like giving up with it, but after just ignoring it and come back later, so you've sort of more idea going into the head? So you sort of like find something or – sort of just suddenly, just something that pop up in your head, you think that's good, and then you try to do it, and work out, so that's give you the satisfaction?

It is obvious in the above quotation that mathematics, and especially problem solving, which is probably the essence of a mathematician's work, is intertwined with a wide spectrum of emotions, both positive and negative. From one side, there is disappointment and frustration, which lead to ontological questions such as "why am I doing this?" and to partial disengagement, and on the other side to satisfaction after an endeavor that leads to the solution of the problem. This process is occasionally extremely tough and often leads to unfortunate outcomes, such as change of subject of study, psychological problems, social disorientation within the department community, etc.

Tracing Calaf's emotional evolution during the course, we have recorded the initial pleasure (with some reference about the difficulty of mathematics) in the first interview. However, in the second interview, even though he initially avoided expressing overtly his emotions, after some encouragement from the interviewer's part, he said the following with some frustration:

> I didn't know how to do it and I tried to find out on the internet, there something, but I still don't know how to find out the number of coset, in the – in the group – I – so I give up on that, so I didn't know how – what to do here, so – need to ask him sometime or – he probably put this – answer on the internet … I'm frustrated cos I just cannot do it ... I try to do it, and then – think

> sometimes you think too long and you thought I just keep up on this, and do the next one, you know what I mean, so that's – you – some sort of frustration.

Cognitive difficulties seem to be directly connected with the affective responses, and as the time progresses this interlock gets stronger, either favourably or unfavourably, depending on the student's progress and performance.

Third Component: Pedagogical Approaches

In the context of this study, the lecturer, a very experienced mathematician, was considered excellent by the majority of students, as expressed in the interviews. This fact is in opposition to the relatively poor results of his students in the coursework and examination. His approach was the traditional "chalk and blackboard", "definition, theorem, proof, lemma" one, enriched with a series of examples and visual images in all stages of the module.

Calaf had strong views and made substantial suggestions about teaching, through which several issues of affection and pedagogy emerged. Two interesting points emerged. In the first excerpt that follows, Calaf expresses his need to communicate his conceptual misunderstandings in a discussion-format

> Instead of three seminars there should be six. So one seminar is for the coursework and one seminar is to discuss and if anyone has any questions, ask them there, and sort of like – small test, like, 10 minutes test and see – summarize whether you understand everything you done from – previous section or something? Rather than just move on to next student, and most of the people turn up in seminars they haven't done – like my friends, they never attempt any question before they attend the seminar, so they just turn up, and then – sometime they just can't do anything, they don't bother to put their hand up and ask any questions, so, I think more – checking more whether they understand it first, before you move on to next section? Otherwise there's no point, move on and if people don't understand it, they lag behind, so yeah.

The second point is related to the one-to-one tutorials with the lecturer in his office. Calaf expresses his fear to visit the lecturer due to the higher level of engagement that such a teaching format requires from a student perspective.

> I think most of the people – just cannot bother to make an appointment to go up and ask – but if you've got a seminar there, and then you be – in front of everyone, like a small group you can talk to them, if you – you see either got different idea to yours, so you sort of like chatting about it first, before you sort of like asking some lecturer? Because the lecturer, you sort of like – quite scary, you think they are quite a scary – to go up and ask them somehow? And like most of the people, in the seminars, are not like – put their hand up to ask any questions, if they don't understand it.

In the above excerpts it is obvious the interlock between pedagogical approaches, the cognitive difficulties and affective responses. Pedagogical activities can have strong effect on students' coping strategies with cognitive difficulties as well as cause (un)favourable development in their emotions and attitudes towards mathematics.

Conclusion

This study is an investigation of the interconnection between cognitive difficulties, affective responses and pedagogical practices in the learning of Group Theory, focusing on the case of Calaf. As the above analysis suggests, there are clear indications according to which cognitive difficulties significantly trigger unfavorable affective responses, which are the cause and effect of (dis)engagement with the pedagogical practices. In a future study, I aim to investigate further this phenomenon, by generalising these results using the entire spectrum of data, from all 13 students. In accordance to the emerging results, there will be an attempt to theorise the Trilateral

Interlock of Learning, as a pedagogical theory by which one could analyse university mathematical learning and examine the study of pure mathematics as a holistic experience.

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