Background Knowledge and Connectedness: The Case of Mathematics

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The New South Wales Pedagogy Model defines three dimensions: Intellectual Quality, Quality Learning Environment, and Significance. The elements of Background Knowledge and Connectedness from the Significance dimension are particularly pertinent to mathematics. The former refers to teaching so that new knowledge is built on existing knowledge, while the latter refers to applying results in ways that have meaning beyond the classroom. This paper argues that mathematics teaching of both early "empirical" and later "invented" mathematics too often has tenuous links to previous knowledge and at best provides superficial applications to real life. It is argued that quality teaching at both levels, while having different emphases, should employ a similar approach to Background Knowledge and Connectedness, namely teaching for abstract-general concepts. However, what constitutes meaningful learning varies with the individual, and invented mathematics may be inappropriate for a large number of students.

A prime factor behind successful student learning has (not surprisingly) been shown to be the quality of teaching (Hill & Rowe, 1998). In New South Wales (NSW), the Department of Education and Training (2003c) argues that recent developments in educational research have shed light on what constitutes quality teaching, and they have consequently established a new model for pedagogy in New South Wales schools (NSW Department of Education and Training, 2003b). This model has as its cornerstone the characteristics of Authentic Pedagogy (Newman & Associates, 1996), which incorporated a focus on both classroom practices and assessment tasks. Two key factors in Authentic Pedagogy are higher-order thinking and connectedness to the real world. The latter is seen as making learning relevant to students and thus providing them with a curriculum which is authentic for them. The Queensland School Reform Longitudinal Study (2001) expanded the Authentic Pedagogy model to what it called Productive Pedagogy, adding areas relating to language and problematic knowledge. The Queensland longitudinal study has in turn provided the basis for the New South Wales model, which comprises three dimensions. Intellectual Quality is the central dimension, with Quality Learning Environment and Significance making up the other two dimensions.

Given the common cry in mathematics' classrooms of "when are we ever going to use this", the elements Background Knowledge and Connectedness on the Significance dimension of the New South Wales model are particularly relevant to mathematics teaching. The NSW Department of Education and Training (2003b) defines these two elements as follows.

> Background Knowledge: Lessons regularly and explicitly build from students' background knowledge, in terms of prior school knowledge, as well as other aspects of their personal lives.
>
> Connectedness: Lesson activities rely on the application of school knowledge in real-life contexts or problems, and provide opportunities for students to share their work with audiences beyond the classroom and school. (p.15)

Both elements can involve connecting to real-life situations. Background Knowledge is backward looking and makes connections with previously existing experience or knowledge. Connectedness is forward looking and makes connections with newly acquired knowledge or experience. To consider how these two elements apply in the case of mathematics, we take a closer look at the nature of mathematical ideas and their teaching.

Booth (1990) describes a paradigm shift in the course of school mathematics, from "empirical mathematics" to "invented mathematics". Empirical mathematics arises from real-life situations and can be explored by contextual investigation. Invented mathematics is developed and extended solely in terms of earlier mathematics. The shift means that we cannot consider the mathematics curriculum as a single entity, but need to examine the implications of Background Knowledge and Connectedness for mathematics pedagogy at the two stages separately.

Background Knowledge and Connectedness in Empirical Mathematics

First, we consider the nature and teaching of empirical mathematics.

The Nature of Empirical Mathematics

Empirical mathematics is the main focus of the early school years, where the relation to concrete experience is hard to ignore. The Number strand is firmly based on the process of counting and grouping objects, and Space and Measurement deal with various properties of these objects. However, empirical mathematics does not stop there. The recent emphasis on numeracy as the mathematics required to participate meaningfully in our current society has resulted in considerable attention being paid to topics such as financial mathematics, chance and data, and graphs of real-life situations right up until the end of secondary school.

The study of empirical mathematics consists essentially of two components.

- Recognising where a particular mathematical concept arises
- Learning how to use that mathematics more effectively

For example, in primary school students learn that equivalent grouping situations lead to multiplication, and learning "times tables" and other techniques enables them to solve problems set in such situations more effectively. The same applies to more sophisticated but still empirical ideas such as ratio, angle and rates of change.

- Ratio: A range of real world situations (e.g., partitioning, percentage swings in elections, odds in betting, proportions in mixing cement or cakes) involve a similar type of comparison between like quantities called a "ratio". Learning how to manipulate ratios abstractly enables one to understand a wide variety of discourse in the press and to carry out one's own calculations in proportion situations.
- Angle: Situations such as corners, slopes, and turns can be identified in the environment, all of which involve the inclination between two lines through at a point—an "angle". Learning to measure angles in abstract diagrams enables one to make more accurate constructions, and learning trigonometry takes this achievement one step further.
- Rates of Change: Again, many common situations involve change (e.g., motion, population growth, temperature change, cost of living) where the rate of change is significant. Representing such situations graphically leads to the ideas of gradient, average rate of change, and instantaneous rate of change, all of which assist in the interpretation of real-life situations. Learning about differentiation provides a means to investigate more precisely change situations which can be modelled algebraically.

All these examples show the power of mathematics: Instead of having to investigate a problem situation concretely, a solution can be predicted using standard manipulations of symbols which represent that situation. Of course, mathematics only provides a prediction, the validity of which depends on how well the symbols really do represent the situation.

Quality Teaching of Empirical Mathematics

The early years. Several recent, highly successful numeracy initiatives (e.g., Count Me In Too in New South Wales, the Early Numeracy Research Project in Victoria, and the Numeracy Development Project in New Zealand) have revolutionised the early years of mathematics teaching. All these projects focus on building concepts on children's own ideas and strategies. Because counting, shape, and length (for example) are so close to the world, the teaching is also linked to children's real-life experiences through the use of teddies, fingers, and counters in Number and real objects in Space and Measurement.

Early teaching of the four number operations often makes use of concrete materials especially designed to teach the desired concepts (e.g., MAB blocks). In one sense these are artificial and not linked to real life. However, we need to be careful what we mean by real life—do we mean the reality of adults or the reality of children? A constructivist approach implies that children should build knowledge from their experience. Since blocks and other puzzles and play materials are a real component of children's lives, these apparently artificial materials really are part of life.

In general, then, the recommended and highly acclaimed approach to teaching empirical mathematics in the early years incorporates a strong emphasis on Background Knowledge. The fact that children and adults alike see the skills and concepts taught at this stage as important supports a similarly strong emphasis on Connectedness.

The later years. Unfortunately, the situation with empirical mathematics at the secondary level is not so rosy. There is strong evidence that many students learn mathematics without making any connection with real situations where the mathematics arises, and that this disconnection explains many of the difficulties they experience in learning and applying the mathematics. Consider again our three examples.

> Ratio: The well-known and common error of using additive strategies rather than multiplicative ones in ratio calculations (Hart, 1982) indicates a weak link between the concept of ratio and real-life multiplicative contexts.
>
> Angle: Many student difficulties arise because the angle diagram does not seem to be easily linked with any real angles. In one study, one third of Year 8 students could not identify angles in slopes and turns (Mitchelmore & White, 2000). Williams (2003) gives an even more extreme example: Her case-study secondary school student successfully made a generalisation about the angle sum of a polygon, but he could not identify the angles of the triangles into which he had divided the polygon.
>
> Rate of Change: White and Mitchelmore (1996) found that many first-year university students approach calculus with a manipulation focus. They do not see the symbols as representing anything, so they cannot use the manipulative techniques they have learned to solve contextual problems. The cause seems to be the way they were taught calculus in school.

Mitchelmore and White (1995) called ideas learned with no link to the contexts in which they arise abstract-apart. They showed that students can only use abstract-apart empirical concepts superficially, that is, in contexts (usually symbolic in nature) which look exactly like the form in which they were taught. In contrast, abstract-general concepts embody the key principles underlying the contexts where the concept occurs. An abstract-general concept comes from a set of real-life contexts and can be applied back to those contexts as well as to further contexts recognised as embodying the same concepts.

Teaching empirical mathematics without linking it to experience seems to be very common. For example, each of 35 Diploma of Education students in the second author's class recently prepared a lesson on empirical mathematics. Thirty of the students chose a non-linked approach. When asked to explain their choice, the students referred to how they had been taught themselves, to the resources available to them to use in the lesson (texts, kits, and the like), and to their desire "not to confuse the students". Noss (2002) suggests that school mathematics has become so decontextualised that it is effectively useless.

> In trying to connect mathematics to what is learnable, we have disconnected mathematics from what is genuinely useful (p.35).

Noss is clearly thinking here of both backward looking connections (Background Knowledge) and forward looking connections (Connectedness), both of which are needed if students are to form abstract-general concepts.

How can Background Knowledge and Connectedness be used in teaching abstract-general concepts? Let us look at our three examples for a third and final time:

- Ratio can be taught by exploring a variety of multiplicative situations, abstracting their common features, practising ratio manipulations, and then applying the skills learnt.
- Angle teaching can be based on identifying the similarities between a variety of real contexts. It is quite possible by this means to teach an abstract-general concept of angle as early as Grade 3, as White and Mitchelmore (2003) have shown.
- An understanding of rates of change via graphs of real-life situations is now seen by many as fundamental in the teaching of differentiation. A leading US college textbook (Hughes-Hallett et al., 1994) devotes a whole introductory chapter to exploring realistic change situations. In Australia similar materials (Barnes, 1992) have been published for high school calculus students, but the approach has not been adopted by most main stream texts.

The key point about building abstract-general concepts in empirical mathematics is that students see how the appropriate mathematical idea is common to all the contexts encompassed by the concept. That is, the key characteristic of learning empirical mathematics is similarity recognition. The similarity is not in terms of superficial appearances but in underlying structure—for example, in recognising the multiplicative nature of the comparison in a variety of ratio contexts or in identifying the arms and vertices in a range of angle contexts. To get below the surface often requires a new viewpoint, as when a student imposes imaginary initial and final lines on a turning object in order to obtain an angle. Recognising similarity is therefore not always easy, and often needs careful guidance from a teacher.

To summarise, in the "empirical paradigm", meaningfully learning that leads to the formation of abstract-general concepts builds on Background Knowledge. Also, if mathematical ideas have been learnt by analysing real-life contexts, Connectedness is a natural extension. In a sense, Connectedness and Background Knowledge are in fact only two sides of the same coin. Quality teaching of empirical mathematics should, therefore, include a strong emphasis on both elements.

Background Knowledge and Connectedness in Invented Mathematics

We now consider the nature and teaching of "invented" mathematics.

The Nature of Invented Mathematics

A common description of invented mathematics is that it is very abstract, meaning that it is totally removed from reality. The essence of this claim is that mathematics is self-contained:

> Mathematics uses everyday words, but their meaning is defined precisely in relation to other mathematical terms and not by their everyday meaning. The syntax of mathematical argument is also precise and concise, with none of the redundancy common in everyday language.
>
> Mathematics contains objects that are unique to itself. For example, although everyday language occasionally uses symbols like x and P, objects like x^0 and $\sqrt{(-1)}$ are unknown outside mathematics.

Self-containment is a crucial feature of invented mathematics. All the many advances of the last few centuries, from group theory to combinatorics, rely on it. It is the lack of reference to any specific context that makes the mathematics applicable to many different contexts and therefore contributes to its usefulness and power.

Historically, mathematics has become increasingly independent of experience as more systems and structures have been invented. Mathematicians look for completion—ways to apply current ideas and results to higher degrees of generality by extending them to larger domains. For example, expressions like 152 and 23 arise in real-world situations involving area and volume. The empirical concept of a power is then applied to expressions for very large and small numbers and to compound interest calculations. To incorporate these ideas into a complete consistent system, however, requires the invention of concepts like zero, negative, rational and irrational powers. Later, this system can be extended further to include powers which are complex numbers. At each point in this extension/completion process, it is crucial that the new objects be related to each other and the previous objects in such a way that they can be operated on without any appeal to any external meaning they might have.

If invented mathematics is self-contained and totally removed from reality, how can its quality teaching have anything to do with Background Knowledge and Connectedness?

Quality Teaching of Invented Mathematics

A large part of invented mathematics consists of rules for operating on mathematical objects and relationships. Some students can learn these "rules of the game" (Sierpinska, 2003) well and may even have some success without any sense of how the rules interconnect. For example, students learn to solve all types of equations—a skill which could gain high marks in examinations. For this reason, it is not surprising that there is often no attempt to apply invented mathematics to anything other than symbolic contexts. Other examples where no application to the real world occurs include symbolic algebraic manipulation in calculus; graphing of polynomial, rational and trigonometric functions; and proving geometric theorems and trigonometric identities.

It would be unfair to suggest that teaching in the higher grades never attempts to apply new mathematical knowledge in some way. Such attempts usually fall into one of two categories.

Artificial exercises: White and Mitchelmore (1996) report on students' responses to an exercise involving a cube of volume 64 cm^3 shrinking at a rate of 96 cm^3 per minute. As one student doing this example (who was obviously well connected) observed, the cube was in its last moments of existence. This type of example has no connection with reality. Instead, it only requires students to strip away a façade of context and uncover the mathematical exercise underneath.

Applications in finance and statistics. For example, the Quality Teaching Program Local Interest Group (2001-03) produced some excellent assessment modules to support the implementation of the Stage 6 General Mathematics Syllabus in New South Wales. In 2001, the modules were algebraic modelling (line of best fit for age and height graphs), financial mathematics, and data analysis. These topics are essentially empirical mathematics, which is why realistic applications could be easily found.

It would appear that large sections of the senior mathematics syllabus deal with abstract ideas where it is too difficult to find related realistic contexts. Does this mean that invented mathematics should be treated as a special case where Background Knowledge and Connectedness are irrelevant? We think not.

Background Knowledge. It is true that self-contained mathematics can only be related to other mathematics, but surely it is still possible to link learning to something meaningful. Otherwise, there is no case for teaching invented mathematics at all! We see at least three ways of building on Background Knowledge to link mathematics to the world of the learner.

An abstract-general approach to teaching invented mathematics can be employed in the same way as in teaching empirical mathematics. In this case, the emphasis is on theoretical similarities (mathematical contexts and their underlying structure) rather than empirical similarities (observable features). For example, consider the index laws. The sequence 64, 16, 4 derived from division by 4 clearly continues 1, ¼, … . The sequence can also be written 4^3, 4^2, 4^1, which to be consistent should extend to 4^0, 4^{-1} … . The structural similarity between the two sequences thus provides the basis for an abstract-general concept of zero and negative powers. The concept is general because the structural similarity is not restricted to powers of 4, but this clearly needs to be learned. In this approach, the rationale for completing a mathematical system can also be expounded.

Most of the invented mathematics studied in schools is either based on empirical mathematics or on invented mathematics which is based on empirical mathematics. Mathematicians do not invent mathematics out of thin air—they build on previous work. So, no matter how advanced a piece of mathematics is, there is at least a tenuous link back to reality. For example, in situations involving complex algebraic manipulation or the graphing of obscure functions, an abstract-general approach can fall back on known previous results and frequently revisit links to empirical concepts. In particular, algebraic manipulation can still be regarded as making generalisations about numbers and algebraic functions as representing a dependency between two variables.

Linking back to the world of mathematics need not be a purely academic affair—it can have a human face. The history of mathematical thought is a rich source of real life stories and excitement, which can place mathematical results in a human perspective. This method of providing stronger ties to real life is also included in the element called Narrative in the New South Wales pedagogy model (another of the Significance elements).

Connectedness. Being able to graph complex trigonometric functions or solve nth degree equations may never have any direct real-life applications for a student in school. Connectedness, as defined by the NSW Department of Education and Training. (2003b, p. 15), therefore, appears unattainable for senior high school mathematics. However, the definition given by the NSW Department of Education and Training (2003a) suggests a broader interpretation of this element.

> Connectedness: Students recognise and explore connections between classroom knowledge and situations outside the classroom in ways that create personal meaning and highlight the significance of the knowledge. This meaning and significance is strong enough to lead students to become involved in an effort to influence an audience beyond the classroom (p. 59).

The key phrase here is "situations outside the classroom … that create personal meaning". There are a number of ways quality teaching could provide personal meaning beyond the classroom for students. We identify three:

Students may see success with algebraic manipulations and trigonometric identities as contributing to their overall mathematical competence and hence to their self-esteem as well as their future career prospects.

The challenge of problem solving can provide personal meaning as students become involved in the processes of generalisation and explanation and then communicate their thoughts and strategies. In a recent discussion with a dozen undergraduate teacher education students of the first author, all agreed that the challenge of problem solving was a great source of engagement and Connectedness.

The creation of self-contained, consistent and complete systems within the world of mathematics can provide aesthetic satisfaction to some students.

We argue, therefore, that Connectedness can be achieved in invented mathematics through qualities unique to the subject.

To summarise, in the "invented" paradigm, quality teaching can and should embrace Background Knowledge and Connectedness, even if these two elements need to be interpreted in a wider sense.

Conclusion

The problem, that many students do not see the point of what they are doing in school mathematics, supports our conjecture that teaching is often disconnected both in the sense of looking back (Background Knowledge) and looking forward (Connectedness). We have proposed that abstract-general approaches to teaching both empirical and invented mathematics could help redress the balance.

Our suggestions for teaching higher level mathematics using challenge, purpose and narrative may, however, only provide a meaning beyond the classroom for the "true believers". Others may find these suggestions put an emphasis on "maths for maths sake" (as one student put it) and so another source of disconnection. There is also the challenge (for the teacher) of how you explain the purpose of many topics in higher level mathematics.

It appears to us that quality teaching as described in the NSW model of pedagogy (especially on the Significance dimension) is that which has been advocated by mathematics educators and teachers for a long time—the provision of a constructivist curriculum appropriate to the world of the student. It may be that well-grounded empirical mathematics is all that most students need in order to function successfully in the world outside the classroom. This argument provides strong support for the various numeracy initiatives in existence and the spirit and structure of the recent mathematics curriculum developments in Australasia. Our conclusion is to advocate empirical mathematics for all, but not necessarily invented mathematics for all.

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