Understanding Geometric Ideas: Pre-service Primary Teachers' Knowledge as a Basis for Teaching

Rebecca Seah
RMIT University
<rebecca.seah@rmit.edu.au>

This paper reports part of an ongoing investigation into aspects of pre-service teachers' geometric knowledge. One hundred and fifty-two Australian pre-service teachers responded to a series of questions that reflect the type of knowledge teachers are expected to know and teach. Analysis of their responses shows that teacher knowledge can be understood through the interplay between individual teachers' formal figural concepts and personal figural concepts. Errors and misconceptions of geometric properties can be addressed by strengthening the link between formal and personal knowledge through visualisation.

Introduction

As one of the oldest disciplines, the learning of geometry is an important aspect of developing intuition in mathematics, spatial reasoning and visualising skills, deductive reasoning, logical argument, and proof (Jones, 2002). Spatial reasoning, the capacity 'to see, inspect and reflect on spatial objects, images, relationships, and transformations' (Battista, 2007, p. 843) are linked to many technological advances and scientific discoveries. It consistently plays a critical role in influencing educational and occupational outcomes of individuals who go on to achieve advanced education credentials in science, technology, engineering, and mathematics (Graham & Pegg, 2011). Despite its importance, there has been scant attention given to research in geometry when comparing to content such as number, algebra and measurement (MacDonald, Davies, Dockett & Perry, 2012). The declining research emphasis has direct and significant impact on the teaching of geometry at all school levels.

To begin, the introduction of new topics in mathematics such as probability, statistics, and computer science has resulted in a reduction of time devoted to the study of geometry in many mathematics classrooms (Mammana & Villani, 1998). Beginning teachers taught under curricula that neglected geometry are likely to overlook the importance of visual and spatial reasoning, as seen in the absence of visual and spatial reasoning mentioned in the Australian Curriculum: Mathematics (Lowrie, Logan, & Scriven, 2012). Indeed, geometry learning today is characterised by memorising the vocabulary and applying formulae in routine arithmetic calculations (Barrantes & Blanco, 2006). There is also a lack of theories to support instructional design efforts. Much of the research into the development of geometric thinking is largely framed within the van Hiele levels (Owen & Outhred, 2006). These studies reported that many students struggle with recognizing geometrical shapes in non-standard orientation, perceiving class inclusions of shapes, visualising geometrical solids in 2D format, and solving problems that require spatial reasoning (Elia & Gagatsis, 2003; Shaughnessy, 1986). Many pre-service and experienced teachers share the same misconceptions about geometry as the students whom they will eventually teach (Fujita & Jones, 2007; Wang & Kinzel, 2014). While the van Hiele levels have provided a general description of the geometric development, they lack the depth needed to inform instructional design (Battista, 2007). Specifically, van Hiele's labelling of 'visual' to the

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lowest level is problematic because visualisation is needed at all levels of development (Jones, 2002).

While the key to supporting meaningful student learning lies in teachers possessing a number of identifiable and differentiable knowledge bases termed mathematical knowledge for teaching (Ball & Hill, 2008), codifying the type of geometric knowledge teachers need is difficult. As a discipline, geometry has grown to include more than 50 different aspects and theories (Graham, Bellert, & Pegg, 2007). Disagreements abound in the aims, content and methods of teaching from primary years to higher education level. As such, 'there has not yet been found – and perhaps there does not exist at all – a simple, clean, linear, "hierarchical' path from elementary to the more advanced achievements of geometry' (Mammana & Villani, 1998, p. 337). All types of geometric concepts appear to develop over time, becoming increasingly integrated and synthesised (Jones, 2002, p. 130). Secure knowledge of two and three dimensional shapes then acts as a conceptual glue that provides coherence and relevance to the learning of more advance geometry (Usiskin, 2012).

Much of the difficulties involved in learning two and three dimensional shapes are caused by a disjuncture between personal geometric knowledge derived from experience and formal geometric knowledge deriving from axioms, definitions, theorems, and proofs. Not so well known is the construct of visualisation and its role in bridging this gap to support learning. Available research suggests that learners tend to be better at drawing a correct image of a shape than providing a definition (Fujita & Jones, 2007). Many learners also have a tendency to make decisions based on figural constraints rather than on formal geometric knowledge (Fischbein, 1993).

This paper reports part of an ongoing investigation into teacher geometric knowledge. The larger study focuses on developing frameworks that can contribute to the design of instructional sequences. The responses of 152 pre-service teachers are considered in order to: (a) determine the gaps between Australian pre-service teachers' personal and formal geometric knowledge; and (b) the role of visualisation in the construction of geometric ideas.

Theoretical Framework

Geometry deals with mental entities constructed through the use of geometrical representations. In the form of points, lines, angles, and shapes, these are not simply representations of actual objects experienced in the world. Rather, they are used in an attempt to take an abstract concept and make it concrete (Phillips, Norris & Macnab, 2010). Geometric representations encompass both figural and conceptual characters (Fischbein, 1993). Figural characters depict properties that represent a certain shape and can be classified as external (embodied materially on paper or other support) or iconical (centred on visual images) (Mesquita, 1998). According to Mesquita, figures can also be determined in terms of 'finiteness' (referring to specific forms) and 'ideal objectiveness' with no reference made to specify its forms. For example, the image ◇ may be considered as a square with the unit of 3 (finiteness) or a quadrilateral with no reference made to its form (idea objectiveness). On the other hand, conceptual characters are *concept image* - the collective mental pictures, their corresponding properties and processes that are associated with the concept (Vinner, 1991). Such an image represents an ideal phenomenon, bound by its *formal concept definition* - a form of words used to specify that concept (Tall & Vinner, 1981, p. 152), and developed through the process of visualising.

Phillips, Norris, and Macnab (2010) found 23 definitions and explicit statements relating to visualisation. They point to a three-fold distinction between physical objects serving as: visualisations; mental objects pictured in the mind; and cognitive processing in which objects are interpreted within the person's existing network of beliefs, experiences, and understanding. Individuals develop their own personal *concept images* and *concept definitions* through experience. They may be referred to as *personal figural concepts* whereas '*formal figural concepts*' refer to *concept image* and *concept definitions* that are based on the axiomatic system (Fujita & Jones, 2007). Problems with visualisation may create disjuncture between *personal figural concepts* and *formal figural concepts.* A learner's first encounter with any geometric ideas is often through the use of objects or geometrical figures. Definitions are used to help form a concept image. Once the image is formed, the definition becomes dispensable or even forgotten (Vinner, 1991). From a didactical point of view, the role visualisation play in the interaction between *personal figural concepts* and *formal figural concepts* may help to understand how geometric knowledge is constructed and thereby inform pedagogical and curricular decisions.

Method

A total of 152 Australian primary pre-service trainee teachers in the third year of a four-year primary teacher education course participated in this study. The participants had undertaken two method courses on number, measurement, geometry, probability, and statistics and were reminded of the geometry topics they have studied prior to the study. Five multiple choice and two short answer questions were presented to the participants and relate to pi, angle, and properties of two and three dimensional shapes. They represent a sample of concepts participants are expected to know and teach. Details of the questions together with the analysis of the data are presented below.

Results and Discussions

The results of participants' correct responses on five multiple choice questions are summarised in Figure 1. No questions obtained 100% accuracy. The best performance was question 2 while the poorest score was question 3.

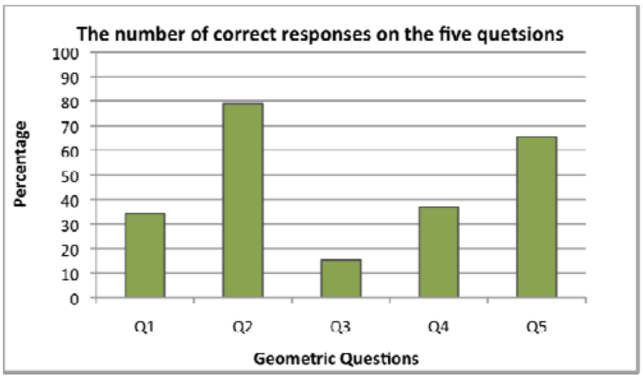

Figure 1. The amount of correct responses on the five questions

The first question asked the participants to select the most appropriate statement about π. It then asked participants to describe an activity that develops an understanding of this relationship. Knowledge of ratio written as a fraction and the relationship between circumference and diameter, through visualisation, can help participants to deduce the correct answer. The results spread across the four options (Table 1). Two thirds of the participants understood the relationship between circumference and diameter whereas 50 participants taught it is related to circumference and radius. Some participants scribbled down $\pi r^2 = C$, $d = r^2$, $2\pi r$, $\pi = r \times 3.14$, $d = 3 \times C$ or πr^2 to help them determine the right answer. Others drew diagrams (Figure 2). None of these participants answered correctly.

Table 1
Breakdown of Responses for Question 1

Questions:	Responses	
1. Select the correct statement about pi	(No.)	
a. Pi is the ratio of circumference to radius in a circle	16	11%
b. Pi is the ratio of circumference to diameter in a circle	52	34%
c. Pi is the ratio of radius to circumference in a circle	34	22%
d. Pi is the ratio of diameter to circumference in a circle	50	33%

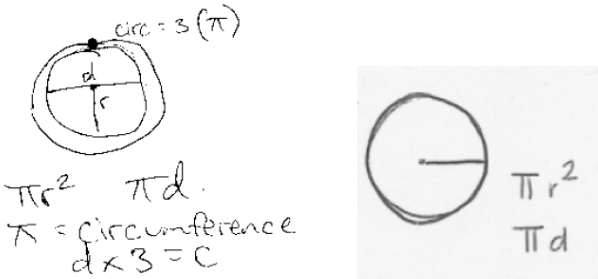

Figure 2. Drawings of two participants.

When asked to describe an activity to develop an understanding of this relationship, many participants mentioned measuring round objects of different sizes and then compare the results to establish the connection. However, a large number of descriptions, as shown below, lacked clarity and showed a lack of *formal figural concept* for the circle.

> Participant A: Pi is the relationship between the radius and circumference. Measure the diameter, half it and use the pi formula to work out the circumference.
>
> Participant B: The diameter is two times the radius, which is half the circumference. Students can measure different bottles and then half the total of the object measured.
>
> Participant C: Circumference is half double the diameter or diameter is half the circumference.
>
> Participant D: Circumference being the distance from one side to the other within the shape, and the diameter being the distance around the shape. I would draw a large circle on the ground and have students use formal and informal or standard and non-standard units to measure the two and see the difference.

During the method course, many pre-service teachers were intrigued by the history of π and methods used by mathematicians to determine the ratio. While almost all could recite π as equal to 'three point one four', few understood that it is an expression of a relationship between the circumference of a circle and its diameter. It would appear that despite the course work, many participants continued to demonstrate a lack of *formal figural concep*t for pi. They did not understand the relationships the formulas they have written sought to express. They also could not infer from the diagrams that since the circumference of any circle is about three times larger than its diameter (based on visualising), the correct answer will have to be 'b' - pi is the ratio of circumference to diameter (based on number understanding). Several participants viewed the diameter as half the size of circumference, albeit confusing both terms.

Question two assessed participants' knowledge of two dimensional shapes. It received the highest correct response (Table 2). Among the 120 correct responses, 77 participants drew figures to obtain the answer. One participant drew and wrote 'equal opposite sides that never meet'. Although the definition is not entirely correct, it showed her attempt to use both her personal *concept definition* and *concept figure* to obtain the answer (Figure 3).

Table 2
Breakdown of Responses for Question 2

Questions:	Responses	
2. David thinks of a regular 2D shape. It has only 3 pairs of parallel sides. The shape could be	(No.)	
a. A parallelogram	17	11%
b. A pentagon	10	7%
c. An octagon	5	3%
d. A hexagon	120	79%

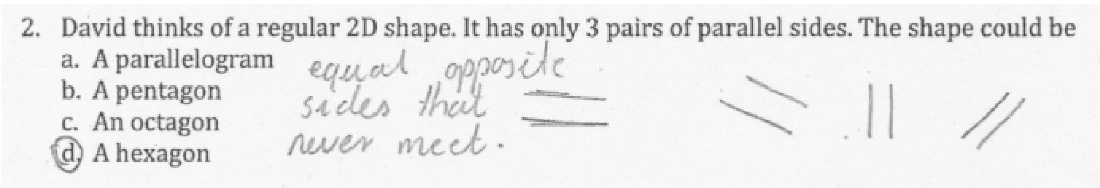

Figure 3. Using definition and figures to deduce the right answer.

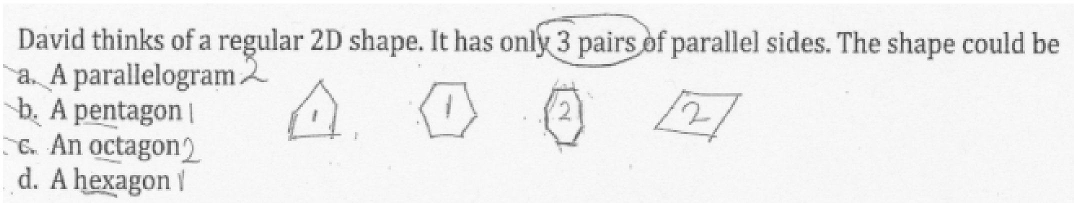

Figure 4. A participant's attempt to draw shapes to solve the problem.

Seventeen (11%) participants answered parallelogram when asked to determine a shape with only 3 pairs of parallel sides. They could have assumed 'parallel sides' as synonym to 'parallelogram'. Those (10 participants, 7%) who chose pentagon could have confused the Greek prefixes of 'penta' and 'hexa' whereas it is unclear how five (3%) participants chose octagon. Among them, one participant drew the four options and attempted to identify the parallel lines (Figure 4). His drawing indicated that he understood parallel lines could be represented vertically (| |), horizontally (=), or diagonally (/ /). However, he could only identify one pair of parallel lines for hexagon base on his diagram. Since the other pair of lines (/ \ and / \) did not rest on the same plane, he concluded that they are not parallel. Because the question asks for 3 pairs, he chose 'octagon' as it has more than six sides. In this case, his visual interpretation of the diagram was incorrect and he did not have sufficient *formal figural concept* for the regular hexagon.

The participants' knowledge of solids was weak and appears to be restricted to prism. Question 3 received the lowest score with 23 (15%) students responding correctly (Table 3). Forty percent of the participants inferred that 'deca' means 10, ignoring the 'do', deduced that a dodecahedron must have 10 faces. Fifty participants may have thought that tetrahedron is made up of triangles with three vertices and so gave the response ‘d’. These participants did not comprehend the Greek origin of these terms. Unlike the hexagonal prism, whose image is easier to be formed in the mind, the participants may not have had sufficient experience with the solids listed in question 3. As such, they were unable to represent three dimensional shapes using two dimensional diagrams. Their responses also suggested a lack of *concept definition* for three dimensional shapes. For question 5, 99 (66%) participants comprehended that a hexagonal prism has 8 faces, 18 edges and 12 vertices. Although 13 participants also knew that a hexagonal prism has 8 faces, they assumed that it has 16 edges instead of 18. This could be due to a counting error or that they were engrossed in the term hexagonal to mean ‘6’. Thirty-nine participants presumed that ‘hexagonal’ meant ‘six’ and chose either ‘b’ or ‘c’.

Table 3
Breakdown of Responses for Question 3 and 5

Questions:	Responses	
3. Select the correct statement about 3D shapes.	(No.)	
a. A dodecahedron has 10 faces	61	40%
b. An octahedron has 6 vertices	23	15%
c. A cube has as many faces as vertices	17	11%
d. A tetrahedron has twice as many edges as vertices	50	33%
5. A hexagonal prism has		
a. 8 faces, 18 edges and 12 vertices	99	66%
b. 6 faces, 16 edges and 10 vertices	10	7%
c. 6 faces, 12 edges and 10 vertices	29	19%
d. 8 faces, 16 edges and 12 vertices	13	9%

An angle is a form of measurement that calculates the amount of turn from one direction to another. Knowing that polygons can be viewed as containing triangles helps understand the patterns for finding the sum of the internal angles for a polygon. Few participants comprehend this idea. When asked to determine the internal angles of regular polygons (Question 4, Table 4), only 55 participants (36%) gave the correct answer.

Table 4
Breakdown of Responses for Question 4

Questions:	Responses	
4. The internal angle of a regular pentagon is		
a. 120°	56	37%
b. 108°	55	36%
c. 110°	35	23%
d. 102°	5	3%

When asked how they would define an angle and demonstrate to a child that the sum of the interior angles of a triangle is always 180 degrees, a number of participants attempted to describe how an angle looks like rather than stating the nature and scope of an angle and its relation to measurement (Figure 5). Also, they could not provide an activity to help a child construct this idea.

> Participant A: An angle is a particular line in which something can be of varying degrees for example the angle could be 90° or 180°.
>
> Participant B: An angle is the degree of two points of radius from a center (starting) point.

Others simply ignored the definition and described how they would get children to draw and measure angles. One participant understood the sum of the interior angles of a triangle is always 180° but drew a triangle with three 90° (Figure 5), demonstrating a lack of *concept image* for triangles.

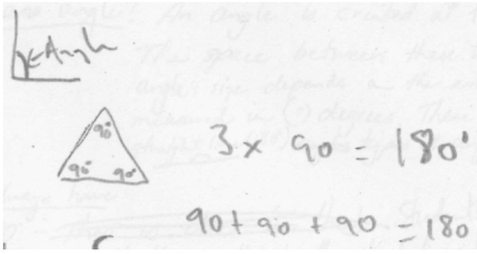

Figure 5. An attempt to define 'angle' by one participant

Conclusion

Teaching for geometric and spatial reasoning requires teachers to have a conceptual understanding of the structures and properties of shapes and solids, their positions in space, and the connectedness between them in the formation of theorems and the learning of other mathematical concepts. The findings indicate that only a small group of pre-service teachers demonstrated sufficient *formal figural concept* knowledge relating to the topics addressed in this study. Many were not ready to teach geometry at the level required of them. While the van Hiele model suggests that these participants are still at level 2 or below and show a lack of geometrical reasoning ability, the constructs of *personal figural concept* and *formal figural concept* provides greater insights into individuals' understanding of geometric ideas.

Individuals' *personal figural concepts* are constructed through experience with various geometric figures and the definitions attributed to these representations. The conceptual characteristics of a figure ares governed by its definition, which in turn is a statement that describes the nature, scope and meaning of a particular concept. For *personal figural concept* to be aligned with *formal figural concept*, well-developed *concept image* and *concept definitions* through visualisation are needed. The findings reveal that many participants' mental images of geometric shapes showed a lack of conceptual understanding. For example, they were able to draw and identify circumference and diameter but did not have the *concept definition* needed to comprehend the relationship between them. They could describe how an angle looks like but were unable to reason using properties of triangles. Many also could not accurately visualise and interpret the figures they had drawn, assuming that diameter was half the length of circumference. They also lacked a *personal figural concept* for three dimensional shapes, suggesting the lack of knowledge to this topic.

Similar to Fujita and Jones' (Fujita & Jones, 2006) findings, regression has happened after participants have completed the method course. One explanation could be that the geometric ideas presented were new to them and have not influenced their underlying beliefs and cognitive processes. This, coupled with their school experience may be the reason why participants can recite formulas but cannot provide the *concept definition*. This study only addresses a limited range of geometric ideas. Further research is needed to investigate the extent of teacher geometric knowledge and classroom practices, and how tasks can be designed to challenge and promote visualisation in the construction of *formal figural concepts*.

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