Proportional Reasoning as Essential Numeracy

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This paper reports an aspect of a large research and development project that aimed to promote middle years school teachers' understanding and awareness of the pervasiveness of proportional reasoning as integral to numeracy. Teacher survey data of proportional reasoning across the curriculum were mapped on to a rich model of numeracy. Results provided evidence of extensive and creative teaching of proportional reasoning in all learning areas. The capacity of such tasks and activities for promoting student numeracy is theorised.

Background

Numeracy is an enabling skill for life and work and means being able to apply mathematics in everyday situations. Many everyday life tasks require proportional reasoning; that is, the capacity to understand and interpret situations of comparison in relative terms (e.g., scaling recipes, currency conversions, calculating discounts). In fact, proportional reasoning has been described as one of the most commonly applied mathematics concepts in the real world (Lanius & Williams, 2003). Yet students' persistent and continued difficulties with proportion and proportion-related tasks are well documented (e.g., Lamon, 2007). An explicit focus on proportional reasoning in all school subject areas, including mathematics, may have great potential for achieving successful development of this essential life skill and therefore numeracy improvement.

Proportional reasoning is being able to make comparisons between the entities in ratio and proportion situations in multiplicative terms (Behr, Harel, Post & Lesh, 1992). The development of proportional reasoning is a gradual process, underpinned by increasingly more sophisticated multiplicative thinking and the ability to compare two quantities in relative (multiplicative) rather than absolute (additive) terms (Lamon, 2005). For example, a proportional reasoner can see that the relationship between the numbers 2 and 10 additively as a difference of 8, but also multiplicatively as 10 being the result when 2 is multiplied by 5. The essence of proportional reasoning is understanding the multiplicative structures inherent in proportion situations (Behr et al., 1992). Students' difficulties in developing proportional reasoning have been attributed to the teaching of mathematics topics in isolation (English & Halford, 1995) and an elementary school curriculum that does not promote multiplicative structures (Behr et al. 1992). There have been calls for change to the way rational number topics are taught in primary school, with greater attention to the active development of students' multiplicative thinking (Behr, et al. 1992; Lamon, 2005; Yetkiner & Capraro, 2009). How this may occur, however, is still unclear.

Theoretical Framework

In Australia, numeracy has been defined as being able to "use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in

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community and civic life" (AAMT, 1997, p. 15). More recently, a much richer description of numeracy has been proposed by Goos (2007) that draws together the myriad definitions of numeracy and simultaneously highlights the absolute necessity of numeracy being a core goal in education, encapsulated into a compact readily-identified triangular figure. The numeracy model has been elaborated elsewhere (see Goos, Geiger & Dole, 2010). It highlights the fact that numeracy is situated within a context, and includes mathematical knowledge, tools, dispositions, and a critical orientation. The model has been found to be extremely useful for analysing the numeracy demands of a school mathematics curriculum (Goos, Geiger & Dole, 2010); to support teachers' curriculum planning (Goos, Dole & Geiger, 2011), to trace changes in teachers' understanding of numeracy (Goos, Geiger & Dole, 2011), in the analysis of the design of numeracy tasks to draw implications for pedagogy (Goos, Geiger & Dole, 2013), and for exploring the role of digital technologies in numeracy teaching and learning (Geiger, Goos & Dole, 2014). In this study, we use the numeracy model to analyse proportional reasoning tasks and activities to theorise their capacity for supporting students' numeracy capabilities.

As stated previously, the essence of proportional reasoning is multiplicative thinking, an awareness of how two quantities are related in a multiplicative rather than an additive sense. The American Association for the Advancement of Science (AAAS) (2001) Atlas of Scientific Literacy identified two key components of proportional reasoning: Ratios and Proportion (parts and wholes, descriptions and comparisons, and computation) and Describing Change (related changes, kinds of change, and invariance). Lamon (2007) outlined central core ideas for proportional reasoning as rational number interpretation, measurement, quantities and co-variation, relative thinking, unitising, sharing and comparing, and reasoning up and down. These two sources highlight the encompassing nature of proportional reasoning and the fact that it is more extensive than simple rules or calculation procedures. In the absence of knowledge of ways to promote proportional reasoning, teachers may revert to skill-based approaches that will hamper students' proportional reasoning development and capacity to use proportional reasoning in complex and unfamiliar situations. Tasks requiring proportional reasoning are a continual stumbling block for so many students in many areas of the curriculum, which suggests the need for a broad-spectrum, multi-pronged strategy for action.

This paper addresses the following research question:

> What is the nature of cross-curricular proportional reasoning tasks in relation to their capacity to promote students' numeracy?

Design and Approach

This project involved approximately 90 teachers from five school clusters comprising secondary schools and their feeder primary schools in geographical proximity. Over the two years of the project, clusters met together eight times, once per school term (four per year). We drew upon the Loucks-Horsley, Stiles, Mundry, Love, & Hewson (2010) framework for designing professional development to guide our approach for project meetings. In between cluster meetings, teachers were to devise learning plans tailored to their own school context, as a result of input from the professional learning seminars and to report back to the cluster at the next meeting. In between professional learning seminars, the researchers visited project teachers in their classrooms, offered support and advice, and assisted with planning and implementing ideas. As such, a design-based research approach (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) was taken in this study as it aimed to investigate and build theory about the enrichment of teachers' numeracy-related subject matter knowledge and practice as well as the improvement of students' numeracy levels. A large corpus of data was collected over this project, and included results from a researcher developed pen-and-paper pre- and post-test diagnostic assessment instrument specifically tailored for this project, to classroom observations, teacher interviews and focus groups, individual student interviews, and teacher feedback surveys.

The data reported in this paper is from a teacher survey, which was administered during the second year of the project (second meeting in Year 2). Teachers were provided with a large sheet of paper containing a table of cells with each curriculum subject area displayed as column headings. Teachers were asked to reflect upon activities and tasks they had implemented in their classrooms that had either been directly focused on promoting their students' proportional reasoning, or opportunities they had seized (teachable moments) for emphasising proportional reasoning to their students. Survey data were analysed three ways. First, the responses were collated into a master list of tasks and activities to give a direct count of the number of proportional reasoning moments described by teachers for each learning area. Second, similar responses in each learning area were collapsed to highlight the different types of proportional reasoning moments that teachers had identified according to each learning area. Third, the proportional reasoning moments were categorised as aligning with particular elements of the numeracy model to give a sense of how proportional reasoning activities might serve to promote numeracy.

Results

A total of forty survey responses were collected, comprising responses from six teachers of Grade 4, nine teachers of Grade 5, nine teachers of Grade 6, twelve teachers of Grade 7, and four teachers of Grades 8-10 (secondary school). Survey return was dependent upon attendees at the workshop at the time. In total, these teachers identified 395 instances of proportional reasoning opportunities, teachable moments, tasks, and activities across the learning areas, including five instances in "Other" areas. In many cases, repetition was seen in the examples provided, so a second level analysis removed repetition, resulting in 284 distinct proportional reasoning moments identified in the learning areas. These results are presented in Table 1.

Table 1 shows that teachers identified proportional reasoning moments in all learning areas, with most counts in Mathematics followed closely by Science. Without accounting for repetition, in the learning areas of Health and Physical Education (HPE), Studies of Society and Environment (SoSE), and The Arts, proportional reasoning moments were identified approximately half as many times as for Mathematics and Science, with the learning areas of English, and Design and Technology approximately one-third as many times as for Mathematics and Science. After repetition had been taken into account, these amounts were similar, except for English and Languages other than English (LOTE) where there was little repetition of examples given by teachers. English examples thus were approximately half the number of examples given for Mathematics and Science.

Examples of proportional reasoning in Mathematics included money, fractions, angles, determining the better buy, using maps, and scale. In Science, proportional reasoning moments included comparing rates for generating electricity, comparing shadows, making predictions based on data, planets, energy, and ramps. Examples of proportional reasoning moments in HPE included balancing diets, ball games and speed, comparing heart rates at rest and after exercise; SoSE examples included devising timelines, latitude and longitude, house plans, paper usage, and percent per capita to population, needs and wants and natural resources; The Arts examples included drawing and body proportions, mixing paint, devising dance steps, perspective drawing, and cartoon drawing; Design and Technology examples included computer usage per country per gender per age group, analysing product packaging, gear ratios, book making, and water quality analysis; English examples included making posters with words in proportion to importance, creating task timelines, analysing ballads, and spatial information in a range of texts. In LOTE, identified proportional reasoning moments included land mass of Japan compared to Australia, time zones, financial exchange rates, and place value associated with other number systems. Examples of proportional reasoning in the 'Other' category included: looking at teacher time on analysing national test data and the amount of time given to planning and developing curriculum, students creating their own study planner, seating plan for the classroom, and staff discussion time on student diagnostic test results.

Table 1
Number (and percentage) of initial identified proportional reasoning (PR) opportunities, teachable moments, tasks, and activities for each subject area, with second analysis removing instances of repetition

Learning Area	Number of initial PR moments identified	PR moments after repetition removed
English	27 (7%)	26 (9%)
Languages other than English (LOTE)	11 (3%)	11 (4%)
Health and Physical Education	43 (12%)	32 (11%)
Studies of Society and Environment	49 (12%)	34 (12%)
Mathematics	98 (25%)	57 (20%)
Science	89 (23%)	67(24%)
The Arts	41 (10%)	32 (11%)
Design and Technology	32 (8%)	20 (7%)
Other	5 (1%)	5 (2%)
Total	395	284

For the third level of analysis, three members of the research team analysed each task separately and then met together to compare classification. Differences in classification were discussed and agreement attained through establishment of guidelines for classification (described below). There was high agreement between researchers with only five instances of differences in classification. Each proportional reasoning moment was considered in relation to the definitions of elements within the numeracy model: *Mathematical Knowledge* (problem solving, estimation, concepts, and skills), *Tools* (representational, physical and digital), *Contexts* (a real-world situation), *Dispositions* (confidence, flexibility, initiative, and risk), and *Critical Orientation* (questioning, hypothesising, interpreting results to make informed decisions).

Although each of the proportional reasoning moments could be categorised as relating to several of the numeracy elements, classification was determined on the basis of emphasis. As such, proportional reasoning moments that were classified as *Mathematical Knowledge* included: finding unknown angles, problem solving using ratio examples, designing fair tests, moon phases, mixing paint, examples that predominantly link to mathematics content and process knowledge. Proportional reasoning moments classified as *Tools* included tasks that predominantly required the use of tools for completion: drawing a pulse rate graph, layers of the earth models, enlarging and reducing images on the computer, and drawing a circle graph.

Table 2
Proportional Reasoning moments categorized according to elements of the Numeracy Model

	Critical Orientation	Context	Mathematical Knowledge	Tools	Dispositions	Total
English	3	6	17	0	0	26
LOTE	0	0	11	0	0	11
HPE	7	9	15	1	0	32
SoSE	12	8	14	0	0	34
Maths	4	17	32	4	0	57
Science	7	23	35	2	0	67
Arts	0	7	18	7	0	32
Tech	4	4	12	0	0	20
Other	1	3	0	0	1	5
Total	38 (13%)	77(27%)	154 (54%)	14(5%)	1 (0.4%)	284

Proportional reasoning moments classified as *Context* were those that specifically located the task within a real context, and included: comparing the proportion of time spent on various themes in a movie, shortcuts to the school oval, orienteering using maps, cooking to create food (as opposed to determining ingredients for fictitious recipes), fuel use on *Mr Brown's* motorbike (as opposed to calculating fuel use for any bike), and exploring why penguins huddle. This category was difficult to determine in some instances as the context provided opportunity for developing a critical orientation, for using tools, and developing mathematical knowledge. However, the authenticity of the context was the determining factor for classification. For example, the calculations for Mr Brown's motorbike related directly to Mr Brown as the students' classroom teacher. This is a real context for the application of mathematics. Some of the proportional reasoning moments listed by teachers clearly linked to the development of a critical orientation, and included: bullying – the victim feels small while the bully looms large; advertising – the size of photos and words for emphasis or persuasion; gambling debt and proportion of club profit; carbon production versus power use and a home audit. In the analysis, there was only one proportional reasoning moment that could be categorised as linking to the *Dispositions* element of the numeracy model, and this was in relation to students creating their own study planner as this was deemed a task where students had autonomy over the outcome, which was very personal to them. It could be conjectured that many of the contexts of the proportional reasoning moments also provided opportunities for development of students' positive dispositions, and this has been found to be the case in other research (Goos, Dole & Geiger, 2011; Geiger, Goos & Dole, 2014), but we surmised that this was not the main focus of teachers' thoughts as they completed this exercise. Table 2 provides a summary of classification of all proportional reasoning moments according to the elements of the numeracy model.

The data presented in Table 2 indicate the high level of potential mathematical knowledge students in this study would be exposed to through engaging in the proportional reasoning tasks identified by their teachers. Table 2 also shows the range of contexts, beyond mathematics in which teachers were incorporating proportional reasoning moments. Of all learning areas, Science appeared to be one that teachers found most contexts for proportional reasoning moments, as well as in mathematics. Surprisingly, teachers identified many proportional reasoning moments in the learning area of English, three of which would potentially promote a critical orientation. Of all subject areas, the learning area of SoSE had the most proportional reasoning moments associated with developing a critical orientation, suggesting that students were engaging in meaningful learning experiences as active and responsive citizens.

Discussion

Survey data suggest strong evidence of a cross-curricular approach by teachers in designing and implementing tasks that promote students' proportional reasoning. Forty teachers nominated 395 instances of proportional reasoning tasks, activities, and learning opportunities across all areas in the curriculum. Whilst Mathematics was the subject area most nominated, this was followed by Science, but this accounted for only approximately 50% of tasks. Tasks and activities associated with subject areas of The Arts, Health and Physical Education, and Studies of Society and Environment were nominated approximately 10% each with just fewer than 10% of tasks located in the subject area of English. Data collection occurred in the final year of the teacher workshops, suggesting that with a greater understanding of the nature of proportional reasoning, project teachers were more responsive to triggers for potential proportional reasoning tasks they could use in their classroom that extended beyond mathematics.

Using the numeracy model to frame analysis of the nominated tasks and activities, we saw richness beyond simply the development of mathematical knowledge, although just over half of the tasks were identified as promoting this numeracy dimension. Just over one-quarter of the tasks primarily were rated as being situated in an authentic context. This means that students were developing and applying proportional reasoning in real situations, in accordance with how numeracy should be developed (Steen, 2001). The most note-worthy outcome of the analysis was that approximately one-eighth of tasks were categorised foremost as relating to the numeracy dimension of a critical orientation. This means that students were being provided with a critical numeracy education that included opportunities to critique, make critical interpretations of mathematical information, use mathematics in a reflective way, and use mathematics to operate powerfully in the world (Stoessiger, 2002). The example of describing how one feels when one is being bullied is a stunning proportional reasoning moment that has a strong social message that would have a profound impact on students.

Clearly, through the high number of counts of tasks that targeted mathematics knowledge in the data, it would appear reasonable to suggest that the students in our project teachers' classrooms were in a much stronger position for developing multiplicative thinking and engaging in processes that comprise proportional reasoning. From the activities listed, we surmised that students would be engaging in rational number interpretation, measurement, exploring quantities and co-variation, relative thinking unitising, sharing and comparing, and reasoning up and down; mathematical processes core to proportional reasoning (Lamon, 2007). Classroom observations that were omitted here due to space limitations provide further evidence of this.

In relation to our research question that guided our analysis here, it appears that cross-curricular proportional reasoning tasks can be grounded in authentic contexts through the nature of the learning area in which they are located, that they have the capacity to promote mathematics knowledge, tools, dispositions, and a critical orientation. As such, the development of proportional reasoning can occur in all learning areas, and as a result, has the capacity to promote students' numeracy. The long history of students' difficulties with proportional reasoning tasks has led to repeated calls for change to the teaching of proportional reasoning in the curriculum (Lamon, 2007; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). Proportional reasoning is generally regarded as something that is located in topics of ratio and proportion, although it has long been identified as something that cuts across subject areas and is most frequently applied in real life (Ahl, Moore & Dixon, 1992; Boyer, Levine & Huttenlocher, 2008; Lanius & Williams, 2003). Taking a cross-curricular approach to proportional reasoning in this project provided teachers with an alternative approach to developing students' proportional reasoning capabilities. The fundamental cause of students' difficulties with proportional reasoning has been levelled at a lock-step mathematics curriculum that teaches topics in isolation (e.g., English & Halford, 1995; Sowder et al., 1998). In this project, teachers circumvented the traditional pathway to rational number teaching, creating new and diverse learning activities that not only have the potential to promote students' proportional reasoning, but also to enhance their numeracy capabilities.

Conclusion and Implications

The research in this paper relates to current educational issues in three ways. First, numeracy is an educational priority on a global scale. The academic debate around defining numeracy has now turned to cross-curricular teaching of numeracy. Our research here shows the creative ways that teachers designed authentic numeracy tasks across all curriculum areas. Second, a focus on proportional reasoning framed within a rich model of numeracy drew teachers' attention to fundamental mathematics content for proportional reasoning that they incorporated into their task/lesson design. Teachers designed a myriad of cross-curricular tasks, showing the pervasive nature of proportional reasoning throughout the curriculum. Teachers' gradual and continued awareness of proportional reasoning highlights its elusive nature. Third, the theoretical frame of this study, combining key research from the fields of proportional reasoning and numeracy, provides a frame for analysis to determine the richness of numeracy tasks whilst simultaneously illuminating essential mathematics content knowledge for proportional reasoning.

In sum, this research has argued that numeracy, as a major goal of education, is essential curriculum, and that proportional reasoning is an essential component of numeracy. The theoretical model highlights core mathematical content knowledge for proportional reasoning whilst simultaneously serving to assess the richness of numeracy practices. Through a targeted approach to numeracy from the basis of promoting proportional reasoning, data presented here suggest that rich numeracy practices can be enacted in the classroom in all learning areas.

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