Specialised Content Knowledge: Evidence of Pre-service teachers' Appraisal of Student Errors in Proportional Reasoning

Mohan Chinnappan
University of South Australia
<mohan.chinnappan@unisa.edu.au>

Bruce White
University of South Australia
<bruce.white@unisa.edu.au>

That the quality of teachers' knowledge has direct impact on students' engagement and learning outcomes in mathematics is now well established. But questions about the nature of this knowledge and how to characterise that knowledge are important for mathematics educators. In the present study, we examine a strand of *Specialised Content Knowledge,* SCK (Ball, Thames and Phelps, 2008) of a group of pre-service teachers in the domain of proportional reasoning. In particular, we were concerned with teachers' knowledge of evaluation of the plausibility of students' claims and errors. Our preliminary results indicate that the participants, as a group, had developed a sense of student error but experienced difficulty in explaining the source of these errors.

Introduction

High school students' engagement with mathematics and their learning outcomes have come under increasing scrutiny from teachers and curriculum policy makers. This issue has received increasing attention against the backdrop of a declining enrolment trend in senior mathematics subjects. While students seem to be showing interest in studying general mathematics subjects, there is an appreciable decline in enrolment in mathematically demanding subjects. In order to arrest and reverse this pattern, it is critical that teachers and teacher educators understand the multitude of factors that could afford or hinder a higher level of student participation than is evidenced hitherto.

The quality of instruction that students receive in their mathematics classroom must surely feature as a significant factor that could impact on students' learning and development of mathematics proficiency. While the quality of mathematics instruction could be analysed from a number of angles, the kind of knowledge that teachers bring to and activate prior to and during teaching can be expected to have a significant influence on students' engagements with mathematics concepts and problem solving skills. In this regard, we argue that, the development of a nuanced understanding of processes and content of mathematics that is taught in our secondary classrooms is a necessary first step in characterising quality of mathematics instruction. As teachers are at the forefront of subject delivery and assessment of student performance, it is imperative that researchers focus on teacher knowledge and how that knowledge impacts on their decisions.

Conceptions of Student Learning of Mathematics- Framework of Schema

In discussions about teaching it is imperative that we unpack notions of student learning and understanding of mathematics. Our conception of student learning is built around the construct of *mathematical schemas*. Mathematical schemas are organised knowledge clusters or chunks of knowledge that are built on and around core mathematics concepts, principles and procedures. Schemas provide an important theoretical tool to facilitate discussions about deep and surface understanding in mathematics. Schemas that are sophisticated can be expected to have more concepts and links between concepts, thus

2015. In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins* (Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia), pp. 157–164. Sunshine Coast: MERGA.

reflecting deep understandings. Students who have large and extensive mathematical schemas are expected to also show fluency in the use of procedures and the use of multiple strategies for problem solving. Drawing on the work of Mayer (1975), Chinnappan, Lawson, & Nason (1998) analysed understanding of mathematics concepts in terms of schemas that have *internal* and *external* connectedness. Thus, in our study of quality of teaching and its relationship to teacher knowledge, we work on the assumption that teachers need to build extensive, deep and well-connected mathematics schemas themselves in the first instance in order to support their students to construct similar schemas. The question is what are the constituents of such schemas for effective mathematics teaching? In order to answer this question, we need to consider the broad categories of knowledge that teachers need to access prior to and during their teaching.

Teacher Knowledge and Teaching Mathematics

In his seminal work on analysing teacher knowledge, Shulman (1986), hypothesised the role of two key components of knowledge that teachers need for effective practice: Content Knowledge (CK) and Pedagogical Content Knowledge (PCK). The identification of CK and PCK strands provided the initial prompt for educators to explore how these two core knowledge bases could support mathematics teaching. Following several lines of inquiry (Ball, Hill, & Bass, 2005; Chinnappan & Lawson, 2005; Walshaw, 2012), there is an emerging consensus that effective mathematics classroom practices are driven by a robust body of teachers' mathematics content and pedagogical content knowledge.

Research interest in knowledge that teachers bring to support learning has gained momentum by recent empirical evidence that teachers' mathematics content knowledge contributes significantly to their students' achievement (Bobis, Higgins, Cavanagh, & Roche, 2012; Senk, Tatto, Reckase, Rowley, Peck, & Bankov, 2012). In broad terms, mathematics content knowledge refers to knowledge of the concepts, principles, procedures and conventions of mathematics, while pedagogical content knowledge involves teachers' understanding of students' mathematical thinking (including conceptions and misconceptions) and representing mathematics content knowledge in a learner-friendly manner.

The pioneering work of Shulman led Ball and her associates (Hill, Rowan, & Ball, 2005; Ball & Hill, 2008) to focus on mathematics teachers and fine tune the knowledge strands that are necessary for teaching mathematics effectively. The outcome of their work was the development of a number of new strands of knowledge clusters for mathematics practice that was collectively referred to as *Mathematics Knowledge for Teaching,* MKT (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008).). We regard MKT as providing a macro schema for understanding and describing teacher knowledge that is critical to their work. Within MKT, there are two main categories of knowledge: Content (Subject-matter) Knowledge and Pedagogical Content Knowledge. The Content Knowledge category was decomposed into Common Content Knowledge (knowledge of mathematics common to most educated adults), Specialised Content Knowledge (specific and detailed knowledge of mathematics required to teach it), and Knowledge at the Mathematics Horizon. In our attempts to better understand teacher knowledge that is necessary for supporting school mathematics, we have been inspired by the above dimensions of teacher knowledge for teaching mathematics that was proposed by Ball and colleagues.

Ball *et al*.'s (2008) conceptualisation of MKT led researchers to develop tasks in order to measure the various components. However, most of this effort has been invested in conceptualising and measuring MKT in the context of primary mathematics. Ball (personal communication) has suggested that there is a need to analyse the character of MKT for junior and senior secondary mathematics. In the present study, we attempt to fill this void by focussing on investigating one strand, namely, Specialised Content Knowledge (SCK) of prospective junior secondary mathematics teachers. SCK is an important strand for two reasons. Firstly, this strand has been shown to correlate with high levels of student learning outcomes, particularly at the primary levels (Ball & Hill, 2008). Secondly, it has been shown that SCK tends to be underdeveloped in most teachers including future teachers of mathematics (Hill, Rowan, & Ball, 2005; Hill *et al*., 2008; Chinnappan & White, 2013).

SCK in Number and Algebra

In discussions about SCK, the mathematics community is concerned with mathematical content that is unique to teaching. This knowledge base includes structuring and representing mathematics concepts, identification of the mathematics that underpins an instructional task and anticipation of different ways students might think about concepts including their misconceptions (Steele, 2013). SCK of a teacher also includes their ability to appraise and analyse unconventional solution methods of their students. In this regard, Ball *et al*. (2008:400) suggested 'looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general' as an important component of teachers' SCK. In the present research, we take up this particular aspect of SCK in the context of a problem that involved proportional reasoning. Our research was guided by the following question: What is the nature of SCK of prospective teachers in the domain of proportional reasoning that involved evaluation of plausibility of student errors?

Methodology

Design

We have adopted a case study design for this study involving groups of pre-service teachers (PSTs) engaging in discussions about a given proportional problem. This design was considered to be appropriate as we aimed to gain an in-depth understanding of a phenomenon - evolving teacher knowledge within groups, as suggested by Yin (2009) and Zevenbergen (2004). The groups of PSTs constituted the units of analysis for the study.

Participants

A cohort of 8 PSTs participated in the study. The participants were enrolled in a Master of Teaching which is a professional Masters leading to a teaching qualification and were then employed in Government schools across South Australia. The participants came from a variety of backgrounds, many had industry experience, some were recent graduates and a number had PhDs. The PSTs had completed two core mathematics methods courses and twelve weeks of professional experiences before the commencement of the study. In this report we provide data generated within one group (4) of the PSTs.

Task

We were conscious that the task that we provided for our PSTs to interact with will engender multiple opportunities to activate their SCK. In a study about teacher preparation, Beswick and Goos (2012) developed a set of mathematical problems that were used to assess content knowledge. From this set, we selected a proportion problem, namely, the Cordial Mixture Problem (CMP) for the present study (Figure 1).

The following question was given to Year 8 students:

Some children are making batches of cordial by mixing together sweet concentrate and water.

Sally uses 4 cups of sweet concentrate and 13 cups of water.

Myles uses 6 cups of sweet concentrate and 15 cups of water.

One student has a misconception and thinks these cordial mixes will have the same sweetness.

Which of the following cordial mixes might this student ALSO think will have the same sweetness?

☐ Aisha uses 8 cups of sweet cordial mix and 26 cups of water.
☐ Carly uses 10 cups of sweet cordial mix and 19 cups of water.
☐ Deng uses 8 cups of sweet cordial mix and 20 cups of water.
☐ Erin uses 10 cups of sweet cordial mix and 28 cups of water.

Figure1. Cordial Mixture Problem

The CMP is regarded as a rich context for the externalization of teachers SCK for the following reasons. Firstly, in examining the solution to the problem, teachers could activate a range of intuitive knowledge about the solution to the given problem as well as examine the mathematics underlying the solution, both of which were regarded as core elements of SCK by Sullivan (2011).

The conclusion by the Year 8 student (Figure 1) that the cordial mixes have the same sweetness suggest that the student have added 2 to both the number of cups of sweet and cups of water respectively. This indicated the use of additive thinking by the student. In contrast, the activation of multiplicative reasoning, in context, would involve comparing the ratio between cups of sweet to cups of water between Sally and Myles respectively (4:13 to 6:15). Such a comparison of relationships would have led the student to the correct conclusion that the ratios are not equal, and therefore, the two cordial mixes are not of same concentration.

In analysing CMP and its solution offered in terms of concepts such as ratio, proportion, additive and multiplicative thinking, we suggest, constitute PSTs' SCK. At the core of this knowledge is reasoning about the multiplicative relationship that exists between base ratios within the given proportional context. That the student had used additive thinking suggests PSTs' awareness of how the student could have reached the erroneous conclusion, a component of their SCK.

Procedure

Participating PSTs were organised into groups of four, they were given a number of questions to complete individually and then asked to discuss their solutions to these problems including the CMP. In sharing their responses, each member of the group was also invited to comment on the problem, identify potential solutions from their students' perspective and issues related to teaching and learning about the given problem. In prompting the participants along the above lines of analysis, our expectation was our PSTs will focus on the key concepts that underpin the different representations and solution paths all of which constitute SCK underpinning the CMP. Each group was allowed a maximum of 30 minutes to complete this activity.

Results and Analysis

We provide transcripts of PSTs' discussions in two excerpts below.

Excerpt 1

PST1: Me to because it was up to us to see the pattern as to see the other pattern, because I've been telling the kids that maths is all about patterns and things,
PST2: Would you see misconception patterns?
PST1: so interesting for us to have to work out where the misconception pattern was.
PST3: So I ended up with 10 cups and 19 cups
PST1: Yes.
PST3: Yeah.
PST4: I interpreted this in a different way I think because...
PST2: I did too.
PST4: ... because the children interpreted two, two groups of concentrated water in the top I think, and down the bottom there I presumed that we had to choose between two of those, because on the top they use the difference of 2 in both sides.
PST2: Two, two on both sides.
PST4: So down the bottom I used the difference of 2 and 2 so I said Aisha the top one and Erin in the bottom one.
PST2: I saw that pattern to, that's the pattern that I saw.
PST3: OK, I think I see what you mean
PST2: I think it's the one where 10 and 19 come from.
PST1: That was my answer.
PST3: Because when you've got a difference of 4 cups of sweet water, you've got 4 and 13, and then it goes 6 15, so then I went 8 17 and 10 19.
PST2: Sorry, I don't understand, you got 4 and 13.
PST4: I just said the...
PST3: 6 and 15 which is the next one.
PST1: You add 2 again.
PST3: Yep, and then the next one up would be 8 17, so you'd have 8 cups which is 17 cups of water.
PST2: 8 to 17.
PST2: And then you'd go to 10 and 19, so I could see that both, both ways could be
PST1: What was your way?
PST4: I just said OK, there was a difference of 2 in both those 2 and 4 to 6, 13 to 15, so then I looked down the bottom and I said, *Right, we have to choose two of those,* so 8 and 10, 26 and 28.
PST2: So you guys looked at it, *what one out of this lot would be the same as those two?*
PST3: Yeah.
PST4: That's not how I interpreted that.
PST2: Which two would be the same
PST1: We picked two others.
PST1: Yeah.
PST1: Oh!
PST1: See I thought that was a separate thing.
PST2: So I think they're both right.
PST3: Yeah, I think they both are right it depends how you read it
PST1: Good old English.
PST4: Mm, it's a bit ambiguous isn't it, not crystal?
PST1: So what would we talk to about that student?
PST2: We'd have to find out why.
PST3: Yeah.
PST4: Why that student thinks
PST1: Well they're not dividing all are they, they're basically not.
PST3: They've just picked a pattern.
PST1: They're not getting a ratio at all.

PST3: That's right, they've just assumed, they've made an assumption that it's going to be the same, maybe you need to be set up to see that it's not the same.
PST1: They're doing differences so you need to go back and show them that they should be doing division with this sort of thing, for ratios

The discussion above highlights some interesting insights into the SCK of these PST's. Initially the discussion centered on the identification and importance of pattern and their ability to identify not only the correct pattern but also an incorrect pattern that the students may have used (turns 3 and 12). However there were two interpretations of the question and so the discussion then focused on interpretation of the question and how easy it was to read the question in a different way than was intended (turns 4, 7 and 11). It also highlighted that it does take some time to see the problem in a different way to how you initially interpret it. Interestingly both were able to answer the question based on their interpretation (turns 16 and 25). However this was a distractor from the intended discussion and made us question the value in not having the researcher as part of the discussion. The discussion then returned to what the student had done and they identified that the student had not used a ratio at all and that they needed to set up a situation where it would not work and that the student would need to use a ratio, although no detail was give how they would do this (turn 46). The discussion appears to show that the students were able to identify the problem and had some idea of what they needed to do but did not have the breadth of SCK required to draw upon to give specific examples of how they would help the students.

The participating PST's were also asked to comment on the effectiveness of the process used – use of CMP as a prompt for externalizing SCK). Below is a short extract of their discussion.

Excerpt 2

PST1: Yep. So I think it's easy to just focus, like just focus on just getting the right answer and like you said, when you're time poor you focus on just trying to get them the basics.
PST2: That's right.
PST1: Instead of stepping back for a second and throwing one of these out there and saying, OK, we've done all these ratios and stuff, let's look at this one.
PST2: Let's move on because the higher-order ones, they have that
PST1: This is how you check they understand it, right?
PST2: Yeah, yeah.
PST1: That they haven't just learnt your tricks and processes.
PST2: That's right.
PST1: That they've
PST3: Yeah, they have to sort of figure out, you know, what they're doing.
PST1: Yes.
PST3: Get an understanding.
PST1: And equally it's, go through this process because if you're just marking a test, that's the wrong answer and you put a cross, something wrong, then obviously you're not going learn anything. If you don't know where they went wrong, if you can't follow it, you can't help them out, so

Comments from Excerpt 2 indicate that PSTs are aware of the need to examine aspects of students' thinking that goes beyond procedural knowledge (turns 47, 56 and 58). This could be evidence of the PST's activation of PCK but one that is reliant on SCK about proportional reasoning. The exchanges also indicate that the PSTs found the process to be useful for them and that they were able to make the connections between the type of problem that they were using and the outcomes that they expect to get. We suggest that the process had made participants think about the SCK that is involved in analyzing CMP although it was at times difficult to distinguish between exchanges involving SCK and PCK.

Discussion

This study that is reported here was motivated by our desire to better understand the state of SCK by a cohort of prospective teachers of numeracy who were enrolled in our teacher education program. We worked on the assumption that by providing opportunities for pre-service teachers to externalise their SCK in informal situations, as teacher educators, we will be in a stronger position to understand the quality of this knowledge. Such data were expected to generate guidelines for supporting their future learning needs in developing their SCK further.

The preliminary data indicate that prospective mathematics teachers' SCK is somewhat tenuous in the particular area of proportional reasoning, an area of mathematics that has been shown to continue to present challenges for both teachers and students (Lamon, 2011; Beswick & Goos, 2012). However, given that the participants are in the early stages of their professional development, there were important signals to suggest that our PSTs have formed precursors of powerful SCK. For example, there is strong evidence that our teachers were keen to explore the ratio schema in which the CMP was anchored.

From a schemas perspective, the CMP acted as an effective prompt for internal and external schemas (Mayer, 1975) about concepts of co-variation, ratio, additive and multiplicative relationships. In this context, understanding of the concept of ratio is part of students' internal schema, whereas deducing the equality of ratios in proportional thinking is a component of external schema. Both schemas are reflective of knowledge that is unique to the work of teachers as suggested in the framework of MKT (Ball *et al*., 2008).

Our preliminary study along this line of inquiry examined SCK in the context of matrices (Chinnappan & White, 2013) among prospective mathematics teachers. The results of that study provided evidence that the quality of representations can be used as a key indicator in studies of SCK. In the present study, we suggest that the analysis of additive vs multiplicative representation of CMP or similar problems by PSTs could be a useful way to extend the current study.

The data that we present here is drawn from four PSTs who were asked to study and comment on the given CMP. As pointed out earlier, the group discussion was conducted in an informal manner with limited intervention from the investigators. While this strategy for data collection was effective, from a methodological perspective, the above arrangement may not have provided an optimal environment to obtain a more complete picture as to the state of the participants' SCK and information for charting its evolution. In a future large scale study, we intend to fine-tune this weakness by involving the researcher in engaging the participants by the use of semi-structured questions to probe the participants both during and post group discussions.

A challenge in the present study was that the conceptualisation of SCK had to be grounded in one specific area of secondary mathematics in order to generate fine-grained data. Within the domain of numbers and algebra, there are numerous areas that are ripe for the exploration of teachers SCK. However, pinning down one area within these strands was problematic for us in order to be able to make general claims.

Capturing the nuances of SCK is also limited by the fact the knowledge is developmental in nature and that any description of this knowledge is only valid at the time of the investigation. Thus future studies should also track the growth of SCK and map a trajectory of the growth by providing different proportional reasoning problems and examine the connection to PCK.

References

Ball, D. L., & Hill, H. C. (2008). Measuring teacher quality in practice. In D. H. Gitomer (Ed.), *Measurement issues and assessment for teaching quality* (pp. 80-98). Thousand Oaks, CA: SAGE Publications.

Ball, D., Hill, H. C., & Bass, H. (2005). Knowing Mathematics for Teaching. *American Educator, 36*(3), 14-46.

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Beswick, K. & Goos. M. (2012). Measuring pre-service primary teachers' knowledge for teaching mathematics. *Mathematics Teacher Education and Development*, *14*(2), 70-90.

Bobis, J., Higgins, J., Cavanagh, M., & Roche, A. (2012). Professional knowledge of practicing teachers of mathematics. In B. Perry, T. Lowrie, T. Logan, A. MacDonald, & J. Greenlees (Eds.), *Mathematics Education Research Group of Australasia: Research in mathematics education in Australasia 2008-2011* (pp. 313-341). The Netherlands: Sense Publishers.

Chinnappan, M., & White, B. (2013). Developments in teacher knowledge for teaching algebra. In M. Inprasitha (Eds.), Innovations and exemplary practices in mathematics education. *Proceedings of the 6th East Asia Regional Conference on Mathematics Education*, Vol. 3 (pp. 202-210). Phuket: Centre for Research in Mathematics Education, Khon Kaen University.

Chinnappan, M. & Lawson, M. (2005). A framework for analysis of teachers' geometric content knowledge and geometric knowledge for teaching. *Journal of Mathematics Teacher Education*, *8*(3), 197-221.

Chinnappan, M., Lawson, M., & Nason. R (1998). Representing the connectedness of mathematical knowledge. In C.Kanes, M.Goos and E.Warren (Eds.), *Proceedings of the 21st Conference of the Mathematics Education Research Group of Australasia*, (pp. 161-168). Griffith University Press: Mathematics Education Research Group of Australasia.

Hill, H.C., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371- 406.

Hill, H. C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L.,, & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Lamon, S. J. (2011). Rational numbers and proportional reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age.

Mayer, R. E. (1975). Information processing variables in learning to solve problems. *Review of Educational Research*, *45*, 525-541.

Senk, S., Tatto, M T., Reckase, M., Rowley, G., Peck, R., & Bankov, K. (2012). Knowledge of future primary teachers for teaching mathematics: An international comparative study. *ZDM – The International Journal on Mathematics Education, 44*(3), 307-324.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *4*(15), 4-14.

Sullivan, P. (2011). *Teaching Mathematics: Using research-informed strategies*. Retreived on January 28 2015, from http://research.acer.edu.au/aer/13.

Steele, M. D. (2013). Exploring the mathematical knowledge for teaching geometry and measurement through the design and use of rich assessment tasks. *Journal of Mathematics Teacher Education*, *16*, 245–268, DOI 10.1007/s10857-012-9230-3

Walshaw, M. (2012). Teacher knowledge as fundamental to effective teaching practice. *Journal of Mathematics Teacher Education*, *15*(3), 181-185.

Yin, R. K. (2009). *Case study research: Design and methods*. SAGE Publications: Thousand Oaks, CA.

Zevenbergen, R. (2004). Study groups as a tool for enhancing students' content knowledge. *Mathematics Teacher Education and Development*, *6*, 4-22.