Problematising Mathematics Education

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We assume many things when considering our practice, but our assumptions limit what we do. In this theoretical/philosophical paper I consider some assumptions that relate to our work. My purpose is to stimulate a debate, a search for alternatives, and to help us improve mathematics education by influencing our future curriculum documents and practice.

Many assumptions are made about mathematics education practice at all levels of education; these need to be identified and questioned as some are no longer appropriate. They relate to our aims, and our conceptions of mathematics, curriculum, teaching and learning, thinking, and assessment; and problematising these is the basis of this paper.

Educational Aims

The taken-for-granted aim of many who teach mathematics is to follow a prescribed curriculum knowing that learners will be assessed summatively. This may seem cynical, but it is also the view of many students, parents, and future employers. It assumes that summative assessment is more important than diagnostic, formative, or self-assessment. While many able students often find such assessment tasks trivial, less-able students learn from such assessment that they are not mathematically capable.

But, putting assessments to the side for the moment, what are our general educational aims and are these relevant for mathematics? Early in my teaching my ideas were influenced by a set of aims published by our teacher organisation (Munro, 1969, p. 1), these were "the urge to enquire, concern for others, and desire for self respect."

These hardly changed when our new curriculum (Ministry of Education, 2007, p. 12) identified five key competencies, "thinking, using language, symbols and text, managing self, relating to others, participating and contributing," which I interpreted as aims. For me both sets of aims are similar and relate to mathematics and to other subjects. For me, thinking includes caring thinking which implies: concern for and relating to other people and living things, and caring for and respecting self. Accepting and interpreting these aims, our task is not to prepare students for the future by teaching for assessment, but: to foster student enquiry, thinking, and self-management. And, as enquiry involves creative and critical thinking, and self-management involves metacognitive thinking, these aims become one key aim, thinking, but it is rarely implemented.

Defining Mathematics

Everyone knows what mathematics is, but it is difficult to define without words such as arithmetic, geometry, algebra, statistics, … This is evident with dictionary definitions, for example: "Mathematics is a group of related sciences including algebra, geometry, and calculus, which use a specialised notation to study number, quantity, shape and space." But, is mathematics a subject formed by a partitioning of knowledge into disciplines, subjects, topics and sub-topics, or, is knowledge holistic? For me all knowledge is interrelated, which implies that when teaching a subject we should emphasise links with other subjects. I see mathematics as the study of relations, relations are sets of ordered

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pairs, and all operations are relations where the first element of the ordered pair is itself another ordered pair, e.g., + = {((2, 3), 5), ((1, 4), 5), …}. This definition from the 'new maths' of the 1960s as presented by Papy and Papy (1963/68, 1971) unifies mathematics; and I believe we need to integrate mathematics with other subjects whenever possible.

Curriculum

Rather than using thinking (or educational aims) as a basis for planning, teachers often begin with the curriculum which has been defined as 'all that is planned for the classroom', where the classroom may be at any level of formal or informal education, or in the classroom of life where learning is determined by and unique to the learner. Within each level of formal education curriculum documents have been written, these include:

- the regional level (national/state/district curriculum; or assessment syllabus);
- the institutional level (school scheme/institution curriculum);
- the individual teacher level (reinterpretation of curriculum to 'suit' the students)
- the learner level (student-constructed learnt curriculum).

The traditional model for curriculum was derived from the 'tree of knowledge' in the Bible (Genesis, 2:17). This metaphor implies a knowledge structure with the trunk as the fundamental ideas, branches as the disciplines, and on to small branches, twigs and leaves.

An alternative is a rhizome metaphor from Deleuze and Guattari (2004/1980). This is based on the notion from botany—a rhizome being a plant with roots that grow underground that sends up shoots all over the place. These shoots seem disconnected, but are linked. What we learn seems like this; we learn little bits first and the connections come later.

Both metaphors imply organising curriculum by structuring knowledge. The tree implies a formal organisation, while the rhizome is less rigid, each shoot represents a topic and only after numerous topics have been explored will the connected structure of our subject emerge. With both metaphors knowledge is the goal—the 'content' rather than the 'context' for learning. With these metaphors the assumption is that some specific knowledge needs to be learnt by everyone regardless of beliefs, backgrounds, and interest.

Considering curriculum more radically involves thinking of knowledge and learning as a complex/living/emerging system. Every aspect is connected and interacts with every other aspect and complexity implies that all connections and interactions are unpredictable. This is based on the work of the Maturana and Varela (1987) who saw living and learning as a complex system, and said 'to live is to learn'; it also explains how other living things learn (e.g., Anathaswamy, 2014; Birkhead, 2012; Chamovitz, 2012). From this perspective all learning is connected, knowing (epistemology) is inseparable from being (ontology); we are always learning, and our task as educators is to enrich the living-learning process. However, what is learnt differs for individuals because of differing degrees of awareness, ability to 'be' in the learning moment, prior knowledge, attentiveness, and ability to make sense of what is said; and learning occurs consciously and subconsciously with each learner developing a unique web of understanding that grows in complexity over time.

Teaching and Learning

From this 'living is learning' perspective, teachers are merely catalysts for learning; but powerful catalysts who influence the direction, speed, and depth of learning. Traditionally teachers based their work on learning theories; and theories like theorems are based on

assumptions that are not made explicit, so practitioners are often not aware of what they are assuming. I have a list of over 160 theories, though none start with *X*; they cannot all be true as they contain elements that are contradictory; Table 1 is an abbreviated version of it:

Table 1
Theories related to learning; what is the X-factor?

Theories related to learning		
Abstraction theory	**J**ob-based learning	**S**elf-directed learning
Behaviourism	**K**inaesthetic education	**T**rial and error
Communities of practice	**L**ecturing	**U**nconscious learning
Drill and practice	**M**astery learning	**V**ocational-based learning
Enactivism	**N**arrative pedagogy	**W**omen's ways of knowing
Friere's critical education	**O**bservation-based learning	**X** ????
Goal-based learning	**P**rogrammed instruction	**Y**in-yang learning
Holistic learning	**Q**uestion-based learning	**Z**one of proximal development
Imitative learning	**R**adical constructivism	

It is useful to think about the learning theory that influences one's work, and wonder what the originator of the theory assumed, and how the theory is interpreted today. I am drawn to theories E and Q in the table—Q is self-explanatory, and E, enactivism, best fits with the view of learning I described when I wrote (Begg, 2013, pp. 81–82) the essence of enactivism is, "learning is living, living is learning, and this is true for all living organisms." From this perspective, I see we and the world as inseparable; we co-emerge—cognition (learning) cannot be separated from being (living). Knowledge is the domain of possibilities that emerges as we respond to and cause changes within our world.

As teachers we know our task is to teach. For me teaching is 'stimulating enquiry by asking questions', not 'telling'; and this is possible. The best mathematics lessons I have seen was in Japan—during the 50-minutes the teacher only asked questions, "What do you think? What do you others think? …" Accepting the cultural concern regarding individualism, the teacher ensured that group work dominated so responses given by individuals were group ones and no 'loss of face' occurred. This epitomised 'teaching as asking, not telling.' Related to teachers 'asking' is learners 'researching and thinking'; thus our task as teachers is to provide researching/thinking activities, but that is not always easy. As Heidegger (2004/1954, p. 15) puts it, "Teaching is more difficult than learning because what teaching calls for is this: to let learn" and I would add: 'and to let think!'

Thinking

Mathematical thinking is often considered as being logical (or critical) thinking; but all the other forms of thinking also seem to me to be relevant within mathematics education. There are many possible classifications of the forms of thinking; my own classification divides thinking into nine slightly overlapping forms, namely: empirical, critical, creative, meta-cognitive, caring, contemplative, subconscious, cultural, and systems thinking.

Empirical Thinking

Empirical or sense-based thinking occurs when we are aware through our senses—seeing, hearing, feeling, tasting, or smelling. It is the dominant form of thinking of young children and the starting point for most conscious thinking. It seems valued by both

western and non-western people; it is important both in its own right and as the basis for other forms of thinking. Being aware through one's senses and remembering is nearly automatic—though by improving one's noticing skills (Mason, 2002) or by becoming more aware (Depraz, Varela, & Vermersch, 2003), the process can become richer.

Empirical thinking involves sensation followed by perception (Restak, 2012); sensation involves detection of information (awareness) using sense organs, and perception is the interpretation/analysis of that information so that it can be remembered and used for some purpose. Interpretation involves constructing meaning, thus empirical thinking is not direct knowing as interpretation is based on prior experience. Sometimes, before a sensation has been interpreted, our body has already reacted unconsciously but intelligently to it, e.g., one cuts one's finger and the body's cells immediately begin to 'intelligently' repair the cut before the brain receives and interprets the cutting sensation.

In mathematics education the main forms of sense-based thinking are visual thinking (interpreting and imagining 2 and 3-dimensional diagrams; using Venn diagrams, arrow graphs, flow charts, Cartesian and statistical graphs, symbols, signs, and gestures; picturing, modelling ideas; noticing (Mason, 2002)); and aural/oral thinking (involving: making sense/interpreting what one hears, and saying what one means).

Critical Thinking:

Critical (rational or logical) thinking is fundamental to mathematics; it involves logic, (which depends on a 'logic' system and initial assumptions). Usually western logic is taken for granted and initial assumptions are made without considering alternatives. Absolute proof is not possible with critical thinking as it depends on assumptions made and the logic system used. One can gather evidence to support a hypothesis; and if all the assumptions are made explicit then a 'relative' proof may be useful—but one counter-example disproves a hypothesis. Words (or symbols) are usually used in critical thinking, but diagrams can also be used (e.g., Venn diagrams in set theory)—proofs are not always possible with diagrams, but diagrams are useful when exploring a problem; though they can mislead (e.g., 'are two straight line that never intersect parallel?' One approach is to draw many examples and conclude that that is true; but it is not true in 3-dimensions).

Western critical thinking has dominated western thinking and resulted in 'advances' in many subjects, but the underpinning assumptions seem often not to be made explicit. This has resulted in 'solutions' to problems without consideration of the consequences (e.g., science problems have been solved without considering the environment; western economics has been based on having more, not having enough; and western philosophy has been concerned with individual rights, not community good).

Creative Thinking

Creative thinking occurs in art, music, literature, but also in other aspects of life when we consider alternatives and ask "what if …?" Our ideas of self, of others, and of things we learnt at home and school depend on assumptions and one can be creative by making these explicit, questioning them, and considering alternative assumptions that other people make. Creative thinking can involve making connections within contexts, finding alternative connections, and finding different solutions to problems in different contexts, or with different initial assumptions (and often assumptions are culturally specific), imagining possibilities, visualising options, conjecturing, modelling reality, designing things, making

and seeing patterns, generalising and specialising, and using analogies. It is important in mathematics at all levels as many problems can be solved in different ways.

Metacognitive Thinking

Metacognition is monitoring one's thinking; it involves: learning to learn, thinking about thinking, reflecting, and self-assessing. It occurs consciously, unconsciously, and automatically. The more one attends to this consciously the more one feels in control. Typical questions one might ask oneself are: Have I done enough? Should I do more? What else could I do? What have I assumed, and could I make a different assumption? Am I happy with this, or do I need to improve it, and how might I improve it?

Caring Thinking

Lipman (2003) wrote about caring thinking, and his ideas fit with aims related to self, family, others, living things, the environment, and culture. Caring is influenced by values, and activities for clarifying values help learners become more aware of (and strengthen) their values. One value is respect, including respect for others with different values. Caring depends on cultural beliefs about 'being', and one may ask, are we all separate; could we exist without other people, other living things, and our planet? Caring thinking involves ethical thinking, emotional thinking and critical thinking. It relates to caring for self, for others, and for the community (local, national and international), and for other living things. In education caring is involved when someone is stuck with a problem—when should one intervene? One person steps in at once to help so the person is not frustrated; another allows the person time to consider alternatives—both reflect caring thinking.

Contemplative Thinking

Contemplative thinking can involve having hunches (intuition), noticing, being still, meditating, and developing awareness. It is associated with religious contemplation and is evident in Shamanic, Vedic, Buddhist, Christian, Islamic (Sufism), and Jewish (Cabalistic) traditions; and in the ways of knowing of numerous indigenous cultures (Abram, 1997; Buhner, 2014; Davis, 2007; Kharitidi, 1996; Wolff, 2001). Contemplation is not emphasised as it was in the past because we emphasise science and critical thinking, but numerous scientists, mathematicians, and philosophers acknowledge its importance (Buhner, 2014). Contemplative thinking builds on empirical thinking and complements critical thinking, thus developing contemplative thinking (or awareness) means developing noticing skills (Mason, 2002), sensory awareness, and openness using analogical thinking (Buhner, 2014). Teachers want students to be reflective, but when asking students to reflect on something they often mean 'think critically about it'. Reflecting from a contemplative perspective means holding an idea in one's mind without processing it.

Subconscious Thinking

Subconscious, unconscious, or bodily thinking is important. Mlodinow (2012) wrote how we are only aware of 5% of what goes on in our brains; our brains unconsciously handle the other 95%. This means our subconscious thinking shapes our empirical (sense-based) thinking, our ever-changing memories, our social interactions, our logic, and our cultural beliefs; how we think about self, others, the world around us; and the assumptions that influence our conscious thinking. According to Davis, Sumara and Luce-Kapler (2008, p. 24) our sense organs register about 10 million bits of information each second but we

are only consciously aware of about 20 of these bits; our subconscious 'thinking' or bodily knowing occurs within the cells of our bodies (and within the cells of all living things) and these cells 'know' what must occur for life—but we are not consciously aware of this knowing. Intuition involves the subconscious becoming conscious. One example of this emerged when a mathematics professor was asked, 'how do you go about solving these difficult problems?' He replied, 'I read the question carefully before going to sleep, then when I wake up I write out the solution.' Thus, mathematics not only involves logical/critical thinking, it also involves contemplative (or unconscious) thinking.

Cultural Thinking

Cultural thinking includes communal/collective and global thinking; and differences arise with people from different cultures. Nisbett (2003) wrote about the different ways that Asians and Westerners think, and indigenous peoples think differently in other ways. These ways are not right or wrong, just different—different starting assumptions, different experiences, different vocabulary, different beliefs and philosophies, different logic systems, different emphasis on nouns and verbs, and so on. An example of this (Nisbett, 2003, p. 141) is when given three pictures—some grass, a hen, and a cow—and asked what goes with the cow? Westerners used an animal/vegetable division; while easterners used a thematic relationship cows eat grass. Two other examples of cultural thinking are:

- from economics—maximising profit is the basis of decisions in some countries, but in other countries environmental considerations are more important;
- regarding self-image—my western-enculturated brain believes I am a self-sufficient individual, yet I cannot exist without the world, the air to breath, and the life forms that provide food; so, am I an individual, or a part of a bigger organism? (And that raises the question, how do people from other cultures see themselves?)

Systems Thinking

Systems thinking is based on notions of complex (rather than simple or complicated) systems. Simple systems are mechanistic, based on cause and effect relationships (A causes B); complicated systems are also predictable though not always obviously (A causes B which causes C which causes … which causes Y which causes Z). Contrasting with these are complex systems; complexity assumes a web of interrelationships with ideas emerging that are not predictable (A, B, C, … all interact but the result is unpredictable as the result emerges from the complexity of the interactions). Systems thinking explains how small catalytic events that are separated by distance and time can cause significant changes in complex systems. Systems thinking techniques are used to study physical, biological, social, scientific, engineered, human, and conceptual systems; and it explains how students who have attended the same class come away with different learning because of slightly different initial ideas.

Thinking in Education

These nine forms of thinking seem to imply a partitioning of thinking into categories, but these overlap and merge. When focusing on a task one does not limit oneself to one form of thinking, one moves smoothly from form to form as the task progresses.

For example:

firstly noticing, sensing, or perceiving a situation (empirical thinking),
then analysing it using logic (critical thinking),
pausing and reflecting (contemplative thinking),
deciding to stop and reconsider (metacognition),
asking oneself, might another assumption be made (creative thinking),
thinking of possible undesirable implications (caring thinking),
being influenced by notions we are unaware of (unconscious thinking),
and so on …

Within education, every topic in every subject at every level is a context for thinking. Thinking in mathematics education is more than listening and remembering, and is enhanced by activities involving: communicating, connecting, problem solving (and problem posing), applying knowledge, using tools (including IT), and reflecting (which links with metacognitive thinking and self-assessment). Ideally such activities need to be included when designing thinking/research focused classroom activities (such as literature reviews, projects, creative activities, discussions, and free-writing). For me research simply means 'enquiry'; and research tasks come in many sizes and in all subjects, and can involve independent or group learning, problem-solving, project work, and tutorials where groups of students with their teacher discuss their plans and progress with projects and receive feedback.

Assessment

The dominant types of assessment are internal (in-school) and external (for awards), but traditionally there are three forms:

- diagnostic before learning to find what students need/want to know;
- formative during learning to find how students are forming ideas/ coming to know;
- summative after learning to find what students know (and understood).

For me, changing our emphasis from knowing to thinking shifts the focus of assessment from summative to formative and to an emphasis on metacognition (thinking about thinking) and life-long learning. The responsibility for all forms of assessment shifts from teacher to learner and becomes self-assessment; which fits with preparing our students for life-long learning. Additionally, when the mode of learning involves research projects then the assessment is unlikely to be whole class, but rather, project based.

Conclusion—Making Changes

I see virtually everything we are doing in mathematics classrooms as needing to be changed! Our efforts to change in the past have been like 'shifting the deck chairs on the Titanic.' What should we do to implement our aims? How can we encourage thinking? Can we change our teaching to 'let learn' and shift from telling to asking? How might we reduce the subject silo effect? What forms of assessment are appropriate? Should we encourage learning to learn rather than learning what is taught? Will learning to learn prepare students for a lifetime of learning?

In the past educational authorities sought 'top-down change' by legislation with new curriculum or assessment policies, but the desired changes were never fully implemented. The alternative is 'bottom-up-change' with small groups of teachers taking professional

responsibility and making numerous small changes. In this situation the role of mathematics educators is to model the ideal changes, discuss them, and encourage and support practitioners in their efforts to change. My aim is that we re-conceptualise:

- teaching as asking, not telling;
- learning as researching and thinking, not memorising;
- assessment as formative and self-assessment, not summative;
- mathematics as being integrated with other subjects, not separated from them: and
- making changes as our personal responsibility, not that of external authorities.

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