Contemplating symbolic literacy of first year mathematics students

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Analysis of mathematical notations must consider both syntactical aspects of symbols and the underpinning mathematical concept(s) conveyed. We argue that the construct of *syntax template* provides a theoretical framework to analyse undergraduate mathematics students' written solutions, where we have identified several types of symbol-related errors. A focus on syntax templates may address the under-developed symbol sense of many tertiary mathematics students, resulting in greater mathematics success, and with the potential to improve retention rates in mathematics.

Introduction

Mathematics derives much of its power from the use of symbols (Arcavi, 2005), but research at secondary level has shown that their conciseness and abstraction can be a barrier to learning (Pierce, Stacey, & Bardini, 2010; MacGregor & Stacey, 1997). Since symbols form the basis of mathematical language, mathematical fluency, like fluency in any language, requires proficiency with symbols, which we call *symbolic literacy*. Under the notion of symbolic literacy lies the notion of *symbol sense* described by Arcavi (1994, 2005), which includes among other components the ability to manipulate, *read through* symbolic expressions, realise that symbols can play different roles in different contexts (this will be emphasised throughout this paper), and develop an intuitive feel for those differences. We have privileged the term *literacy* in order to convey the idea of mathematics as a language of discourse (Usiskin, 2012) that can take place in oral or written form.

> Mathematics is, among its many other attributes, a language of discourse. It is both a written language and a spoken language, for – particularly in school mathematics–we have words for virtually all the symbols. Familiarity with this language is a precursor to all understanding. (Usiskin, 2012, p. 4)

The notion of *symbolic literacy* encompasses the understanding of what we believe to be one major feature of mathematical development (see also Usiskin, 1996; Rubenstein and Thompson, 2001) and is at the core of our current studies (e.g., Bardini & Pierce, 2015). However, for the purpose of this paper we will focus on its written aspects since this feature is the nature of our data.

Quinnell and Carter (2012) note that while inaccuracies in spelling and word usage in everyday English text usually do not prevent the reader from understanding the text, even small errors in the use of mathematical symbols may have a major impact on making meaning of the written mathematics. Take, at a very basic level, the common error of omission or misuse of parentheses. Students do not always recognise, for example, that $(-1)^2$ and -1^2 have different meanings or that $\left[2+6\times\sqrt{4}\right]^2$ and $\left[(2+6)\times\sqrt{4}\right]^2$ do not mean the same and do not have the same value.

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At university, not only does mathematics become much more symbolic, but its writing is more subtle and requires increased *flexibility* from the reader; we anticipate that many students may have difficulty with the new and more intense ways in which symbols are used at university, referred to as *symbol load* in our previous work (Bardini & Pierce, 2015). In a study involving first year university physics students (Torigoe & Gladding, 2007), it was found that students' performance is highly correlated to their understanding of symbols. We anticipate that similar outcomes apply to other mathematical sciences at university, with the consequence that students may not understand the mathematical content as well as they did at school, potentially leading to a decrease in positive affect, which in turn might discourage enrolment in further mathematical subjects.

As a first step towards investigating these larger questions our aim is to provide tools that enable us to better examine students' understanding and use of mathematical symbols and therefore gain a better comprehension of students' symbolic literacy. In the following sections we will present the frameworks underlying the construct of such tools and show how these enable us to gain a fine-grained description of students' understanding of symbols, in particular through their writings.

Theoretical Framework

Skemp (1982) identified two levels of language, distinguishing between the surface structures (syntax) of mathematical symbol-systems and the deep structures that embody the meaning of a mathematical communication—the mathematical ideas themselves, and their relationships.

Serfati (2005) also provides us with an epistemological approach to mathematical notations that takes into account both the syntactical aspect of a symbol and the underpinning mathematical concept(s) conveyed. Note that we will use the term *symbol* throughout this paper, but in this particular instance the term *sign* could be thought to be more appropriate (the limitations of this paper do not allow us to fully discuss this).

Following Serfati's work we can analyse symbolic expressions by considering each of their components and distinguishing three features:

- the *materiality*. The materiality of a symbol focuses on its *physical* attributes (what it looks like), including the category the symbol belongs to (a letter, a numeral, a specific shape, etc.).
- the *syntax*. The syntax of a symbol relates to the rules it must obey in the symbolic writing. This includes the number of operands for symbols standing for operators but also the *legitimacy* of a symbol being juxtaposed to adjacent symbols.
- the *meaning*. The meaning of the symbol is the concept being conveyed (the representation of an unknown, of a given operation, etc.). Meaning for Serfati is that commonly agreed by the community of mathematicians and it does not refer to a person's individual understanding.

To work with a mathematical symbol, one not only has to recognise it in the text (i.e., through its materiality), but to select the right meaning and appropriate syntax in that context, which sometimes has to be interpreted very locally (e.g.,, the symbol '–' in front of a number, between matrices).

Since we are considering students' symbolic literacy from a writing perspective, the syntactical aspect of mathematical expressions plays a substantial role. Sherin (1996) provides an alternative yet closely related framework to Serfati's notion of *syntax* (originally called *combinatorial syntax* in Serfati 2005) for the syntactical aspect of mathematical expressions. In a study with third semester engineering students, Sherin asserted that particular arrangements of symbols in physics equations express particular meanings for students, allowing them to understand the equations in a relatively deep manner. He introduces the concept of *symbol patterns*, which can be understood as templates for the arrangement of symbols. As the students developed physics expertise, they acquired knowledge elements that Sherin (1996) refers to as *symbolic forms* consisting of two components: a symbol template, for example □ = □, and a conceptual schema. The schema is the idea to be expressed and the symbol template specifies how that idea is written in symbols, so that students learn to associate meaning with certain mathematical structures. Sherin's *symbolic forms* bear resemblance to Tall's (2001) *procepts*.

Methodology

The research described in this paper formed part of a preliminary study of the extent to which first year university mathematics students experience symbol overload due both to increased symbol intensity and their lack of familiarity with the symbols themselves. This preliminary study led to a current three-year project on this matter funded by the Australian Research Council.

The participants (21 in total) were a tutorial class of first semester undergraduate students enrolled in Calculus 1 in a major Australian university. Data was collected during normal weekly tutorials in which students completed worksheet exercises and problems based on their current lecture topics. It was the normal practice in these tutorials for students to work, standing in pairs or groups, writing their solutions on whiteboards. The tutor moved around the tutorial room, checking students' progress, pointing out errors in the students' solutions and suggesting appropriate methods when students were unsure how to proceed. As observers, the authors of this paper were able to photograph students' solutions but were not able to converse with them as this could disturb the progress of the students' work. These photographs constituted the data. The students' written solutions captured in these photographs were analysed in order to look for evidence of facets of their symbolic literacy through identified errors in particular. This paper focuses on students' solutions to some exercise questions during one of two tutorials relating to complex numbers (tutorials 7 and 8, end of April 2014).

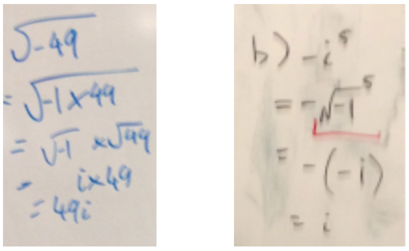
Results and Discussion

The student solutions included below have been selected as representative illustrations of typical errors made by the students. These will be analysed by both considering Serfati's (2005) notions of materiality, syntax, and meaning and by incorporating the idea of symbol template (Sherin 1996) that we will rather call *syntax template* so to ensure coherence with Serfati's framework. For most of these students, the week of tutorial 7, which had included two lectures on the topic, was their first encounter with complex numbers. The materiality, that is, the *shapes* of the symbols and their combination with other symbols, were all familiar from school algebra but some of the syntax and meaning were not. For example while students were already familiar with Latin letters standing for unknowns, variables, etc., the letter i in a complex number takes a very precise and new meaning. Also, while square roots were so far applicable to positive numbers, here the syntax of *square root* is expanded to include negative numbers.

It was clear that every example in these practice exercises involved complex numbers so students were focusing on applying their new learning. In these circumstances it seems that errors in their established templates for syntax were exposed. Illustrations of such errors come from students' responses to questions in tutorial 7 and are detailed in what follows.

Illustration 1

Question 1 of tutorial 7 asked: "Simplify the following, expressing your answers in Cartesian form $a + ib$ where a and b are real numbers. (a) $\sqrt{-49}$; (b) $-i^5$". Figure 1 shows the solution to those items given by two groups of students.



a b

Figure 1. Answers to Question 1a and 1b.

Figure 1a shows the solution to Question 1a, where students have omitted to take the square root of 49, resulting in an incorrect answer of $49i$ instead of $7i$. We conjecture that this is not a mere case of *having forgotten* (a common response from students and, we believe, a likely reply from these students had we had the opportunity to query them). We believe that one potential source for this error lies in the difference in meanings that a same materiality of a symbol (here '$\sqrt{\ }$') conveys. So far, students have always *decoded* '$\sqrt{\ }$' as meaning the process *take the square root of* along with its specific properties (the same that apply for exponents). With the introduction of the imaginary unit i with the property $i^2 = -1$, '$\sqrt{-1}$' is no longer considered as a *square root of* or, in other words, that its syntax template is of the form $\sqrt{\quad}$, rather it has to be considered as one *block* $\sqrt{\square}$, and perceived as the symbolic representation of i. Figure 1a shows that the students did this successfully, moving from $\sqrt{-1}$ (third line) to i (fourth line). However, it seems that the students at the same time see the whole sentence '$\sqrt{-1}\times\sqrt{49}$' with the syntax template $\sqrt{\quad} \times \sqrt{\quad}$ and apply (wrongly) the properties for square roots, in particular the one that says that if you multiply two square roots (provided the arguments are the same) then they *cancel out*.

In Figure 1b, the students have incorrectly evaluated $\sqrt{-1}^5$ as $-i$ instead of i. Similarly to students' response shown in 1a, they have correctly *translated* the symbol i into the *symbol block* $\sqrt{-1}$, but this seems to be what causes them to move incorrectly from the second line to the third. Having considered $\sqrt{-1}$ as one element, this might have led students to now view $-\sqrt{-1}^5$ with the syntax template *negative to an odd power is negative* and too quickly applying this rule to what the *block* $\sqrt{-1}$ means (this thinking is apparent from the usage of brackets in '$(-i)$'), leading to the incorrect intermediary result '$-(-i)$'.

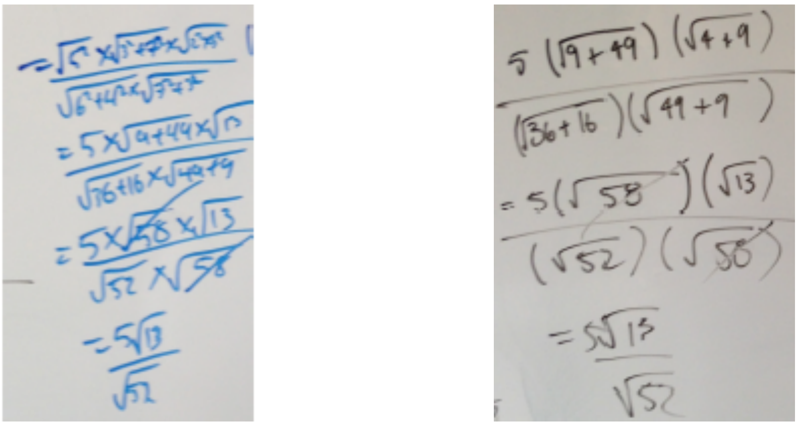
Illustration 2

Equally interesting to looking at students' answers is analysing the questions themselves, since being symbolically literate also means, in some sense, to appropriately read and make meaning of what is asked, including having to sometimes decode *hidden messages* in the stimulus.

In Question 4b of the tutorial, students were asked: "Find the modulus of the following complex numbers without multiplying into Cartesian form: $\frac{-5i(3-7i)(2+3i)}{(6+4i)(7+3i)}$,,

Question 1, for the tutorial, required students to flexibly navigate between different meanings of a symbol with the same materiality ($\sqrt{\ }$); that is, to easily translate square roots in terms of imaginary units as well as to use the fact that $i = \sqrt{-1}$. In order to successfully answer Question 4, students must, on the contrary, *lock* the meaning of *i* as a symbol standing for the imaginary unit, without further considering its intrinsic property. Should the students replace *i* by $\sqrt{-1}$, that would indeed lead them to the numerical dead end $\frac{-135\sqrt{-1}-25}{46\sqrt{-1}+30}$. In fact (and as a consequence), the whole sentence, for example, 3–7i is now to be seen as a whole. This is reinforced by the prompt in the stimulus *without multiplying into Cartesian form*. Because *i* has the same syntax as any other letter, one might be tempted to apply the distributive law to (3–7i)(2+3i). Whilst applying the distributive law eventually leads to the expected answer (5/2), underlying the question is the need to work with properties of the modulus of complex numbers (the modulus of the product of complex numbers). The need to *see the sentence as a whole* goes beyond the syntactical interpretation just described (i.e., to not apply algebraic manipulations as one would for syntactically similar expressions). This specific item required going (or at least was intended to go) beyond the syntax template '□ – □*i*' and rather view it as a complex number. It is the context (complex numbers) and certainly the mathematical conventions (except if we are in electricity or electronics courses where *j* stands for the imaginary unit) that guide the interpretation of the syntax. More importantly, it is the context that will signal an efficient approach to finding the appropriate answer. This will be discussed below.

Figure 2 shows the approach taken by two groups of students in Question 4. First of all, let us note that students have indeed recognised each element of the expression as a given complex number as they then immediately start by (correctly) applying the definition of the modulus of complex numbers and their properties. They then carry out the correct mathematical procedures to finally provide numerical answers. The students have certainly failed to notice that 13 is a factor of 52, hence not recognising that the fraction 13/52 is equivalent to ¼, yet their answer is mathematically correct. So where is the problem (if any)?



a b

Figure 2. Unsimplified numerical answers.

At a basic level, we expected students to question their approach: is it reasonable, at this stage of their mathematical experience that the question posed is meant to test the ability of manipulating square roots? Also, students seem to blindly manipulate mathematical expressions, without ever questioning their meaning in context (certainly a magnitude of a complex number can take any numerical positive value, but we expected that students would have used the meaning of the original expression–the modulus of the complex number–to try and make sense of their final answer and, therefore, prompt them to simplify the result). But the issue is less about students providing a mathematically valid answer than it is about them having not fully unravelled the subtleties of the question, including reading *beyond the mere syntax* of the mathematical expression provided. In fact, a successful and more efficient solution to the problem requires interpreting the modulus of complex numbers without necessarily having recourse to the Pythagorean formula, and to rather interpret the meaning of, for example, $|3-7i|$ (and all other expressions) in the geometrical sense. Having done so, students would have been able to *cancel out* pairs of moduli (e.g., $|3-7i|$ and $|7+3i|$) and come up with a very much more efficient solution. We see in this example to the complexity of being able to navigate between meanings of expressions *with same materiality* and we anticipate this is even more problematic if students are too often exposed to drill types of exercises, as these students' responses seem to suggest.

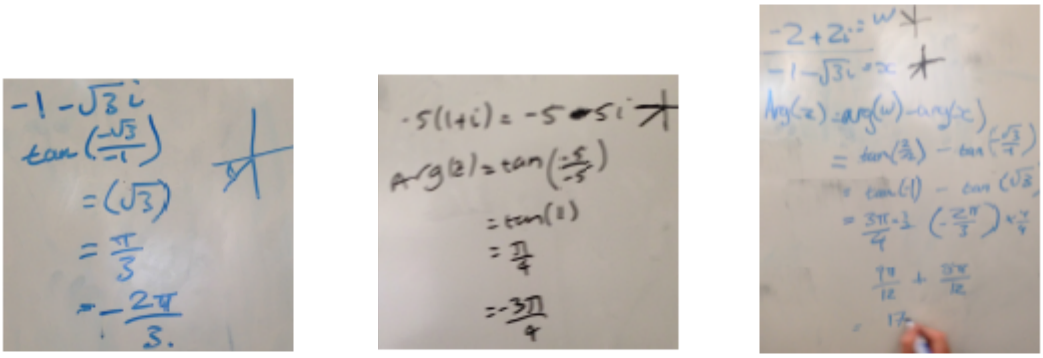
Illustration 3

Question 5 of the tutorial asked:

"Find an argument θ, where $-\pi < \theta \leq \pi$, for the following complex numbers. For part (iii), use facts about the argument of a product or quotient, rather that simplifying the expression.

(a) (i) -5	(ii) $1+i$	(iii) $-5(1+i)$
(b) (i) $-2+2i$	(ii) $-1-\sqrt{3}i$	(iii) $\frac{-2+2i}{-1-\sqrt{3}i}$"

Taking a generic complex number, $a+bi$, the appropriate symbolic form for the argument θ is $\theta = \tan^{-1}\frac{b}{a}$ (or $\theta = \arctan\frac{b}{a}$), taking into account, of course, the signs of a and b to determine the appropriate angle. The students whose solutions are shown in Figures 3a, 3b, and 3c have each obtained the correct values for the arguments but all three show flaws in their written responses.



a b c

Figure 3. Incorrect symbol template and disregard for meaning of equals sign.

First is the confusion between the tangent and the inverse operation, leading to an inappropriate use of the syntax template for the tangent of an angle. In fact, as tautological as it may seem, one has to note that when considering a syntax template, not only are we considering it as a template (much as equation editors in document processing software) but also the syntactical rules that apply for each of its elements (precisely what Serfati 2005 called *combinatorial syntax*). It is almost as if each of the *empty boxe*s of the template come with a precise *domain* (in the functional sense). So, for example, the symbol $\sqrt{}$ has the template of the form $\sqrt{\quad}$, where the empty box has to be filled by a number (given or unknown). Interestingly enough, some of these *domains* evolve or change depending on the mathematical context where they are used. In the case of $\sqrt{}$, we have seen that, while we remain within the set of real numbers $\mathbb{R}$, only positive numbers can fill the empty box. Once we incorporate the set of complex numbers $\mathbb{C}$, this restriction is no longer valid and the *template for the same materiality'* (loosely described) then gains an extended *domain*. Students' responses in Figure 3 suggest that students do not consider the syntax of expressions when it comes to the *domain* of the template for tan , not realising that *tan* prompts for its argument to be an angle. It would seem that students should be encouraged to verbalise their symbolic expressions, stating orally that the argument is equal to *the angle whose tangent is* (see Figure 1b) and linking this with the appropriate syntax template.

The students' syntax, if read aloud, does not make sense. They seem to be working out *the answer* without expecting that the symbols they are writing convey a meaning to the reader. Their responses suggest they are using '=' to say "and then I did something (the reader must guess what that was) and the result is*"*. This and *the result is* meaning of the '=' sign dates from primary school and is deeply set in students' thinking. The notion of expecting symbols to have meaning and a habit of checking the meaning of the symbols used is an aspect of working mathematically that needs to be cultured at all levels: primary, seconday, and tertiary. The work shown in Figures 3a, b, and c suggests that students have thought about the meaning of the symbols, indicating the size and position of the angle locating the complex number on the Argand plane, but have only taken this into consideration once they had finished their calculations.

Conclusions and Implications

The examples that we have chosen illustrate the value of following Serfati's (2005) approach to analysing mathematical notation that takes into account both the syntactical aspect of a symbol and also the underpinning mathematical concept(s) conveyed.

First, careful consideration of materiality is important for both teachers and students. The choice of letters and the form of the symbol act as a cue to the student in making choices about efficient solution methods (Illustration 2). Teachers need to help their students learn to recognise such cues and students need to take a moment to consider the makeup of each symbol rather than relying on unthinking recognition of syntax templates.

Secondly, in the examples shown above it is clear that the students' focus is on the new aspects of working with complex numbers. We can see them trying to employ new syntax templates but either failing to look at familiar materiality in a new way or, in a combination of new and old, misapplying old syntax templates. The notion of syntax templates can help teachers identify likely causes for students' errors and provides a way of talking about the structure and meaning of symbols where in one context students need to recognise a symbol as indicating a process but in another identifying a combination signifying a concept (Illustration 1)(Tall et al., 2001).

Thirdly, Illustration 3 highlights what happens when students do not expect mathematics to be read with logical meaning. Here the lack of conventional templates, where '=' indicates that the expressions prior and following are equal, leave the reader guessing as to the meaning intended.

Mathematical literacy (Usiskin, 2012) may be promoted through contemplation of syntax templates by both teachers and students.

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