A Framework for Teachers' Knowledge of Mathematical Reasoning

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Exploring and developing primary teachers' understanding of mathematical reasoning was the focus of the *Mathematical Reasoning Professional Learning Research Program.* Twenty-four primary teachers were interviewed after engagement in the first stage of the program incorporating demonstration lessons focused on reasoning conducted in their schools. Phenomenographic analysis of interview transcripts exploring variations in primary teachers' perceptions of mathematical reasoning revealed seven categories of description based on four dimensions of variation, establishing a framework to evaluate development in understanding of reasoning.

"Citizens who cannot reason mathematically are cut off from whole realms of human endeavour" (Kilpatrick, Swafford, & Findell, 2001, p. 1). Mathematical reasoning is a broad term encompassing several important aspects such as induction, deduction, abduction, and adaptive reasoning and has been variously defined in curriculum documents (*Australian Curriculum Assessment and Reporting Authority*, 2013). Whilst mathematical reasoning is mentioned in some frameworks of teachers' mathematical content knowledge, there is currently no theoretical framework that addresses this important capability in particular. The development of a framework that could be used for determining teachers' awareness of aspects of mathematical reasoning is the focus of this research paper.

The framework below was developed through phenomenographic analysis of transcripts of interviews with 24 Australian and Canadian primary school teachers. The interview was conducted after the first stage of *Mathematical Reasoning Professional Learning Research Program* (Loong, Vale, Bragg, & Herbert, 2013). Results based on the initial interview for a sub-sample of the current study showed that primary teachers were not confident in defining "reasoning" or they provided evidence of confused or incomplete understanding (Loong et al., 2013), indicating the need for a framework based on teachers' actual perceptions of reasoning in order to capture a more nuanced expression of their understandings.

Researchers from the phenomenographic research tradition describe learning as a change in the way a student perceives the object of learning (Booth, 1997). They gather data about the participants' current perceptions and focus on interpreting the descriptions given by them. The focus of phenomenographic research is the participants' perceptions of the phenomenon under investigation rather than the researcher's perception of the phenomenon. The researcher seeks to probe the participants' existing perceptions, to describe how the participant sees the phenomenon. The aim of phenomenography is to prepare an outcome space consisting of categories of description structured by the dimensions of variation (Marton, 1986) emerging from the data. Facilitating learning is about changing those aspects of the phenomenon that are the focus of the learner, so the development of a framework of teachers' perceptions of mathematical reasoning will emphasise different aspects of the phenomenon. The simultaneous discernment of a greater number of aspects implies a more sophisticated level of understanding, with more aspects of a concept in the foreground of a person's awareness.

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The Study

Phenomenography was employed for this investigation as it "is a research method adapted for mapping the qualitatively different ways in which people experience, conceptualise, perceive, and understand various aspects of, and phenomena in, the world around them" (Marton, 1986, p. 31). The product of a phenomenographic investigation consists of an outcome space of categories of description, structured by dimensions of variation. Dimensions are the means by which the categories are distinguished one from another and are based on empirical evidence in the data. Commonalities and differences between the categories are defined by the particular values of the dimensions that relate to a particular category.

The participants in this study were 24 primary school teachers, 17 from four schools in Victoria, Australia, and seven from one school in British Columbia, Canada. The mathematics teaching experience of the teachers ranged from less than two years to over 25 years, with half of them having seven or less years of teaching experience. The grades of the teachers ranged from Kindergarten (4–6 year-old children in Australia and 5–6 year-old children in Canada) to Grade 6/7 (11–13 year-olds). The demonstration lesson was taught by one of the researchers to Grades 3 and 4 classrooms in Victoria and a Grade 3/4 class and a Grade 6 class in Canada with other participant teachers observing.

The intent of the demonstration lesson and post-lesson discussion was to expose the teachers to opportunities to notice variation in mathematical reasoning, in both the reasoning used by the children and also the reasoning privileged by the demonstration lesson teacher. Teachers had the opportunity to read the detailed plan for the demonstration lesson that documented anticipated student responses and possible teacher actions such as enabling and challenging prompts. During the observation of the lesson they were able to see and hear the reasoning of the children. Immediately after observing the lesson the teachers engaged in a discussion with the other observers, thus giving them an opportunity to experience variation in the ways in which reasoning was perceived by other members of the group. In the two weeks following the demonstration lesson, the teacher participants trialled the lesson in their own school. After this experience, they were interviewed about the children's learning, reasoning, their experience of teaching the lesson, and their reflections on their own knowledge and practice of teaching reasoning. The transcripts of the interview following the trial of the lesson in their classroom provided the data for the phenomenographic analysis. The *Mathematical Reasoning Professional Learning Research Program* provided a context in which teachers could reflect on their perceptions and understanding of reasoning and experience variation in mathematical reasoning.

Developing the outcome space

The outcome space, consisting of an ordered list of categories, depicts the relationship between the categories of description based on the dimensions of variation (Marton, 1986). The outcome space can be seen in Table 1. The author pooled all responses and each response was examined to discern the way mathematical reasoning was perceived. Meaning statements emerged from repeated reading of the transcripts, which provided insights into the different ways the participants saw reasoning. The meaning statements relating to the same perception of reasoning were grouped (Herbert & Pierce, 2013) to establish the initial *categories*, which in turn led to the initial *dimensions*. The categories, dimensions and transcripts were re-examined several times by the different researchers in the team to refine and confirm the categories and dimensions.

The results reported in this paper is an outcome space of seven categories of perceptions of mathematical reasoning held by the teachers in this study (Table 1), with each category determined by one or more values from each of the dimensions: Audience; Purpose; Presentation; and Type of reasoning. The categories indicate a growing awareness of more values of the dimensions and hence a more inclusive understanding of the complexity of mathematical reasoning. The hierarchy based on the theme of expanding awareness of reasoning can be seen in the outcome space (Table 1) and supported by the values of the dimensions taken for each category, for example, the values shaded in Figure 1 illustrate the manner in which the values of the dimensions delineate one category from another and establish the hierarchy of the outcome space. Only one category is described in detail here, but similar descriptions exist for the other categories (Herbert, Vale, Bragg, Loong, Widjaja, submitted).

Table 1:
Outcome Space of Primary Teachers' Perceptions of Mathematical Reasoning.

Category	Perception of mathematical reasoning
Category A	Reasoning is perceived to be thinking.
Category B	Reasoning is perceived to be communicating thinking
Category C	Reasoning is perceived to be problem solving
Category D	Reasoning is perceived to be validating thinking
Category E	Reasoning is perceived to be forming conjectures
Category F	Reasoning is perceived to be using logical arguments for validating conjectures
Category G	Reasoning is perceived to be connecting aspects of mathematics

Category D: Reasoning is perceived to be validating thinking

Audience	**Purpose**	**Presentation**	**Type of reasoning**
Self	Recount	Verbal	Adaptive
Others	Compare/ Contrast	Symbolic	Inductive
	Make choices	Diagram/Written	Deductive
	Explain	Gesture (action)	Inferential
	Argue step-by-step		
	Articulate reasons		
	Justify		
	Hypothesise		
	Generalise		
	Prove		
	Evaluate		
	Connect		

Fig 1. Values of dimensions in Category D.

Teachers presenting this perception referred to the importance of children justifying their explanations of their thinking. Figure 1 illustrates those values of the dimensions that are discerned in this category to show that it includes the additional purposes of explaining, articulating reasons and justifying. This category is also associated with the expectation that the justification may be presented either verbally or diagrammatically. The following quote introduces a particular idea about reasoning that illustrates Category D and distinguishes it from the other categories:

> … a lot of students can write it down and think to themselves, "Yep that's fine, I know it, I've done it, it's satisfactory". But actually explaining it to someone else, whether myself or a peer and being able to justify that …

Discussion and Conclusion

The outcome space demonstrates the variety of primary teachers' perceptions of reasoning. The range of perceptions of reasoning reported in this paper suggests that mere inclusion of reasoning as a proficiency in the Australian or Canadian curriculum is not sufficient to bring about change in teachers' practice and professional learning. So it is to be expected that professional learning will be required to enhance teachers' knowledge of reasoning and the teaching of reasoning. It will be important to track changes in perceptions over time to determine the extent of enactment of the curriculum.

This phenomenographic analysis revealed seven perceptions of mathematical reasoning held by the teachers who were interviewed after observing and trialling a lesson designed to focus on reasoning in a primary classroom. The result of this analysis is an outcome space of primary teachers' perceptions of mathematical reasoning, accompanied by descriptions of the categories relating to the variation in perceptions (Herbert et al., submitted). Categories in the outcome space show expanding awareness of mathematical reasoning from explanation of thinking to include forming conjectures, justifying and validating conjectures, and making connections between and structuring mathematical ideas. The framework developed from the phenomenographic analysis of the data collected for this project provides a way of assessing teachers' growth in their understanding of reasoning.

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