Why You Have to Probe to Discover What Year 8 Students Really Think About Fractions

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Many researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions. This paper examines the persistence of whole number schemes among 14 year-old students who appear to have successfully mastered routine algorithms for working with fractions. Uncovering whole number thinking among such students is therefore difficult, and is illustrated through the use of several probing interview tasks, revealing quite different forms of whole number thinking. These forms of thinking can give correct answers also making it difficult for teachers to identify incorrect thinking about fractions. Representations of fractions using number lines can assist in identifying and correcting such thinking.

Hunting (1986), Streefland (1984), Bezuk (1988) and Kieren (1980a; 1980b) have suggested that difficulties experienced by children solving rational number tasks arise because rational number ideas are sophisticated and different from natural number ideas and that children have to develop the appropriate images, actions and language to precede the formal work with fractions, decimals and rational algebraic forms.

The research reported above has focused quite properly on difficulties experienced by young children when they first encounter rational numbers. Relatively, little research has taken place with older students. Like Hannula (2003), this paper focuses on difficulties with fractions experienced by older students. Many of these students appear still to use whole number thinking which Behr, Wachsmuth, Post and Lesh (1984) defined as "making separate comparisons of numerators and denominators using the ordering of whole numbers" (p. 332). Hart (1981) also notes that "a fraction of course involves two whole numbers which have to be dealt with as if they were irrevocably linked" (p.69). The ratio between numerator and denominator is, according to Hart (1981), the "irrevocable" link.

This research reported in this paper was conducted during 2002 and 2003 as a part of a professional development program *SINE* (Success In Numeracy Education) designed to assist teachers of mathematics in Years 5-8 (with students from 11 to 14 years of age). The Fraction component of *SINE* consists of two Fraction Interviews and two Screening Tests (Pearn & Stephens, 2002a; 2002b; 2002c; 2003).

An initial Fraction Interview (Pearn & Stephens, 2002a) was developed to ascertain students' knowledge and/or their misunderstandings about fractions. The authors decided it was imperative that teachers use an interview that would enable them to observe and interpret their students' actions as they worked on a set of tasks set in a variety of fraction contexts. From a student's verbal and non-verbal behaviour, an interviewer can infer something about the student's internal representations, thought processes, problem–solving methods, or mathematical understandings. The Fraction Interview tasks included contexts such as discrete items, continuous lengths, fraction walls, and number lines.

Because some teachers expressed concern about the time taken to administer the Fraction Interview, a paper-and-pencil Fraction Screening Test was designed with tasks that paralleled the Fraction Interview tasks, wherever possible, including contexts such as discrete items, continuous lengths, fraction walls, and number lines.

Developing a First Probing Interview for Fractions

Given the poor performance of some Year 7 and 8 students on particular items of the Fraction Screening Tests, we looked for evidence of thinking strategies that had lead to these poor results. While paper-and-pencil tests show patterns of strengths and weaknesses, they generally fail to disclose the kinds of thinking used by students, and their "peculiar" algorithms, that sometimes give correct answers.

We conjectured that inappropriate whole number thinking strategies were being commonly applied to fraction problems. For example, during a video-taped interview (Pearn & Stephens, 2002) Robert, a Year 7 student, who otherwise showed sound conceptual and procedural understanding of the fraction tasks, involving for example, addition of fractions and equivalent fractions, gave an unexpected explanation. When asked why he had decided that $\frac{2}{3}$ was larger than $\frac{3}{5}$, Robert said: "From two to three (comparing numerators) is one and from three to five (comparing denominators) is two, so $\frac{2}{3}$ is bigger than $\frac{3}{5}$". Robert's explanation is an instance of whole number dominance as Behr et al. (1984) define it. In other instances of "whole number dominance", students typically calculated the difference or 'gap' between numerator and denominator to compare fractions. Between the 2 and the 3 in $\frac{2}{3}$, for example, they said the 'gap' is 1, while $\frac{3}{5}$ has a 'gap' of 2, making $\frac{2}{3}$ the larger. Whole number thinking includes other strategies where students deal with numerators and denominators individually, ignoring the ratio connecting numerator and denominator. By contrast, we define *multiplicative thinking* as those strategies which preserve the fundamental ratio between numerator and denominator.

We looked for evidence of inappropriate whole number thinking among secondary students who had been taught fractions for at least the previous four years. A Probing Fraction Interview (Pearn & Stephens, 2003) was designed to investigate thinking about fractions among twelve students in Year 8 (14 years old). It had three parts:

Part A: Nine baseline tasks including recognition and completion of equivalent fractions, ordering fractions, placing simple fractions on a number line, and matching fractions with decimals.

Part B: Four questions intended to disclose inappropriate whole number thinking

Part C: Four scenarios for students to critique inappropriate whole number thinking

Parts A and B were given to all students. Only if students successfully completed these were they given Part C. An example of a Part A task is as follows:

Point to the cards with pairs of fractions: $\frac{2}{5}\quad\frac{2}{3}\qquad\frac{1}{4}\quad\frac{2}{8}\qquad\frac{1}{3}\quad\frac{2}{3}$

Interviewer: Which of these pairs of fractions are equivalent? (Pause for response.)

How did you decide?

In Part B, tasks included completing equivalent fractions, deciding on the input required for a given output from a "fraction machine", and choosing the larger of two fractions. For example, the interviewer pointed to two cards, one showing three-fifths and the other two-thirds, and asked: "Which is larger?" After the student responded, the interviewer asked: "How did you decide?"

The scenarios of Part C used actual student responses embodying inappropriate whole number thinking. The scenarios asked students to identify the misunderstanding used by the student and where possible to give an example which would help the student in the scenario. One such scenario presented "Robert's thinking" as described earlier. A second scenario had "Jennifer", comparing $\frac{3}{5}$ and $\frac{2}{3}$, arguing that $\frac{3}{5}$ must be larger because it has a larger numerator and denominator than $\frac{2}{3}$. After being given these scenarios, students were asked: "Do you agree with the way this student got the answer?" and "Can you think of an example that you could use to explain why this method does not always work?"

Results from the Probing Interview

The students interviewed using the Probing Fraction Interview fell into three distinct groups. The first group, Proficient multiplicative thinkers, not only used algorithms correctly and efficiently in dealing with fractions, but were able to deal confidently with fractions of discrete sets and of continuous quantities. Most importantly, these students were able to challenge inappropriate whole number thinking and could provide convincing counter examples. Presented with Jennifer's scenario, one student replied that "this (thinking) would lead you to say that $\frac{4}{8}$ was greater than $\frac{1}{2}$, - which we know are the same. Jennifer's method doesn't work."

A second group may be called *Residual* whole number thinkers. These students used algorithms correctly to represent fractions in a range of equivalent forms. They were generally confident in dealing with fractions of discrete sets, but not surprisingly confused a fraction number with a fractional part of the number line when the line was longer than one unit, as reported by Hannula (2003) and Novillis-Larson (1980). However, these students tended to revert to inappropriate whole number thinking when faced with new or unfamiliar fractional problems. They were less confident in challenging inappropriate strategies but, when prompted, some were able to use alternative methods to check their reasoning.

A third group may be called *Default* whole number thinkers. These students more or less consistently, treat numerators and denominators in ways that ignore the fundamental ratio between numerator and denominator. Some questions in Part A and all four questions in Part B identified clearly those in the third group whose thinking is best described as *default* whole number thinking. These students were not asked to attempt Part C of the interview.

It was in Part C that *Residual* whole number thinkers experienced greatest difficulty. Unlike *proficient* multiplicative thinkers, they were unable to mount a challenge to the scenarios presented. They were often unable to draw upon alternative strategies to complete the task embodied in the scenario. To further investigate the thinking of this middle group, the authors decided that the Probing Fraction Interview needed to be extended further.

Findings from Further Probing of Whole Number Thinking

To further investigate *residual* whole number thinking among Year 8 (14 year old) students, the authors interviewed eight Year 8 students from an all-boys metropolitan secondary school where the Mathematics classes recently had been re-organised after the mid-year exam into three ability groups: upper, middle, lower. The students interviewed all came from the middle ability group, and were nominated by their teachers as being near the top of that group. Six girls from a co-educational metropolitan secondary school were also interviewed. These girls were also from classes that had also been organised according to test results and they were classified as being in the middle ability group.

All students were given four tasks from Part A of the Probing Fraction Interview to confirm that they were not *default* whole number thinkers. Thirteen of the fourteen students were successful with the four tasks from Part A. Some incorrect responses, based on faulty arithmetic, were given by individuals for different tasks. Table 1 shows the four tasks and examples of both correct and incorrect responses. The student who gave the incorrect response to Task 2 (see Table 1) was eliminated because he appeared to be a *default* whole number thinker. However, the remaining students were deemed by the authors not to be *default* whole number thinkers.

Table 1
Tasks and Examples of Responses

Task	Sample correct response	Sample incorrect response
1. This is three-quarters of the lollies I started with. How many lollies did I start with?	One-quarter is 2, two-quarters are 4 … so whole is 8	The whole was 18 because “three-quarters is one more than one-half. One half would be 12 so a whole is 18.”
2. $\frac{2}{5}\ \frac{2}{3}\quad \frac{1}{4}\ \frac{2}{8}\quad \frac{1}{3}\ \frac{2}{3}$ Which of these pairs of fractions are equivalent (have the same value)?	One-quarter is equivalent to two-eighths as “1 goes into 4, four times, and 2 goes into 8, four times”.	One-third and two-thirds were equivalent because “the bottom is the same” but when asked if there were any other equivalent fractions said: “Then probably one-quarter and two-eighths. You can change it.”
3. Point to the equivalent fractions. $\frac{3}{10} = \frac{21}{\square}$ Write a number in the box so that the fractions are equivalent?	Typically students used the rule to complete these equivalent fractions. e.g. How many times does 3 go into 21 (7) then multiply 10 by 7.	All were successful except A. C. who decided that $\frac{3}{10}$ was equivalent to $\frac{21}{60}$ due to a faulty algorithm.
4. Point to the cards: $\frac{1}{2}\quad \frac{1}{4}\quad \frac{1}{10}\quad \frac{2}{5}$ 0.5 0.25 0.1 0.4 Match each fraction with the equivalent decimal.		A few students from both schools matched two-fifths with 0.4 because it was “just the one left”.

After completing the initial four tasks the 13 students were then presented with three questions from a new Part D which asked them to compare three pairs of fractions: $\frac{3}{5}$ and

$\frac{2}{3}$, $\frac{3}{5}$ and $\frac{3}{4}$, $\frac{3}{5}$ and $\frac{5}{8}$; and to use a number line, marked 0 to 1, to illustrate their thinking, for one or more of these pairs.

Some students from both schools compared the pairs of fractions using equivalent fractions using common denominators while others compared the pairs of fractions using their decimal equivalents. For these students who converted fractions to decimals or percentages the task became difficult as they were not always aware of the appropriate equivalent of five-eighths. The students who attempted to use equivalent decimals or percentages appeared not to have any other strategy to use if they could not remember the percentage or decimal equivalent value instantly.

Illustrations of Distinct Forms of Whole Number Thinking

In Part D, several forms of whole number thinking became apparent when students were asked to choose the larger of two fractions. Students used either "*gap thinking*", or "*comparing-to-a-whole thinking*", or "*larger-is-bigger thinking*". Examples used by eight students are presented here. The first two categories were easy to define.

Gap thinking was evident when Student 1 was asked to compare $\frac{3}{5}$ and $\frac{5}{8}$, and said: "Three-fifths is larger because there is less of a gap between the three and the five (in the first fraction) than the five and the eight (in the second fraction)." Student 7 suggested that two-thirds was larger than three-fifths because "two-thirds had 'one spare' therefore it was bigger than three-fifths".

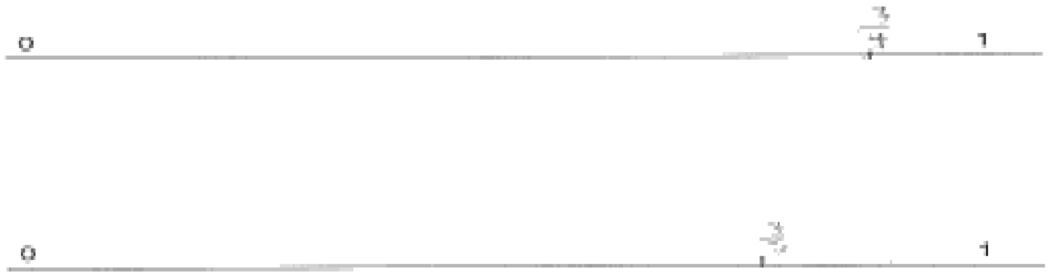
Comparing-to-a-whole thinking was demonstrated when, asked to compare $\frac{3}{5}$ and $\frac{2}{3}$," Student 2 said that two-thirds was larger because: "Three-fifths is *two numbers away from being a whole* and two-thirds is *one number away from being a whole*". Student 5 suggested that: "two-thirds was larger than three-fifths because three pieces are bigger than five (pieces)."

Larger-is-bigger thinking took several forms. Initially, to compare $\frac{2}{3}$ and $\frac{3}{5}$, Student 3 correctly converted the fractions to $\frac{18}{27}$ and $\frac{18}{30}$, but concluded that $\frac{18}{30}$ was larger because: "30 was larger than 27". A different form of *larger-is-bigger thinking* was evident when Student 3 converted $\frac{3}{4}$to $\frac{18}{24}$ and $\frac{3}{5}$ to $\frac{12}{20}$ and then compared both numerators and denominators to decide that $\frac{18}{24}$ was larger than $\frac{12}{20}$ (c.f. Jennifer's thinking). If he had used his earlier thinking on $\frac{3}{4}$ and $\frac{3}{5}$, (same numerator, different denominators), he would have chosen $\frac{3}{5}$.

Using Number Lines as a Further Probe

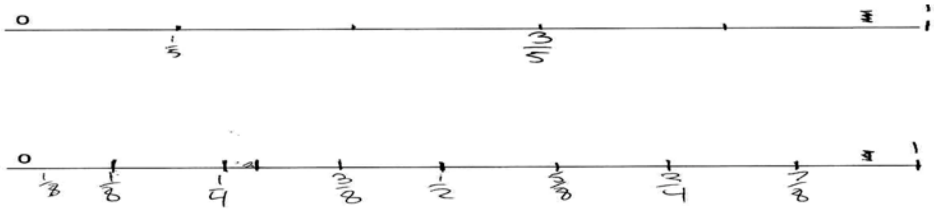
After responding to the previous tasks of comparing fractions students were asked to place a pair of fractions on the number lines marked zero to one. Some students from both schools just placed the fractions on the number lines without using any referents to other known fractions, for example, one-half.

In some cases, students placed the fractions on the number line to reflect their previous responses. For example, Student 6 randomly placed the fraction three-quarters close to one on the number line then placed three-fifths the same distance from three-quarters as she had placed three-quarters from one. This was because, “three-quarters is only *one away from a whole* and three-fifths is *two* a*way from a whole* (gap thinking)”.This was a similar response to one given by a student from the other school.



Student 5 divided a number line into four reasonably equal parts and marked correctly the position for three-quarters. However when asked to show three-fifths she divided the line into six parts and marked the midpoint as three-fifths. Further, to show the fraction five-eighths, she marked another number line with eight marks, that is nine divisions, and marked the fifth one. These difficulties prevented this student from explaining her thinking

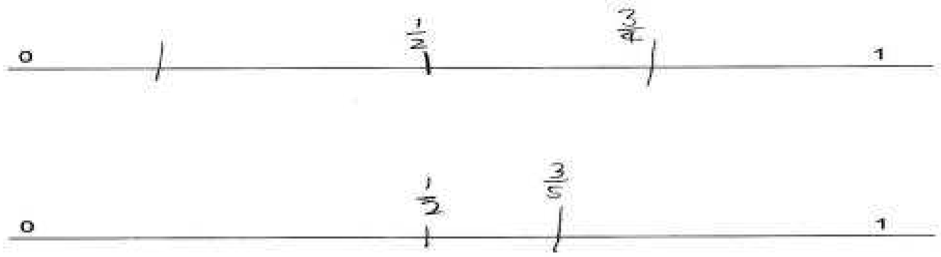
Student 2 also applied *comparing-to-a-whole thinking* to three-fifths and five-eighths arguing that the first must therefore be bigger because *it is only two away from being a whole*. Student 2 was then asked to think about two-thirds and three-quarters. His response: “I think that they are equal. Not just because they are *one away from being a whole*. This (three-quarters) is 75% and two-thirds is about 75%”. He did not, when asked, have a strategy for checking. To further probe his thinking, Student 2 was asked to compare three-fifths and three-quarters using a number line. On the number line (below), he marked in three-fifths by dividing the line into fifths and marked the numbers one-fifth and three-fifths. On the second number line, he marked in one-half, one-quarter and three-quarters by eye (quite accurately). He then concluded that three-quarters was bigger than three-fifths. He reiterated that three-quarters was 75% and used a calculator to show that three-fifths was 60%.



Secondly, to compare three-fifths and five-eighths, Student 2 proceeded to subdivide the second number line from quarters to eighths, by eye (quite accurately). Having placed all the eighths, he then said: “Five-eighths is bigger. It is a bit ahead of three-fifths. My old approach (comparing-to-a-whole) doesn’t work”. To conclude, the interviewer asked him to consider one-half and four-eighths. He said: “My old approach would say that one-half is bigger, but they are the same”.

To compare three-fifths and two-thirds, Student 4 said: “Both go into 15, and then represented two-thirds as ten-fifteenths and three-fifths as nine-fifteenths. To compare three-fifths and five-eighths he initially said that “three-fifths *is bigger by one*”, using *comparing-to-a-whole thinking*. Student 4 then converted both fractions to the same denominator ($\frac{24}{40}$) and ($\frac{25}{40}$), and revised his first answer, saying that $\frac{5}{8}$ was larger.

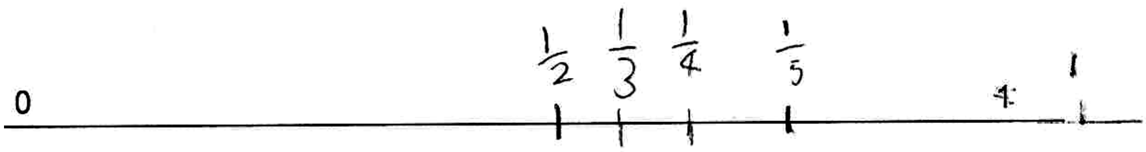
To compare three-fifths and three-quarters, Student 4 correctly converted both fractions to twentieths concluding that three-quarters was bigger. He was then invited to use number lines to compare these two fractions. He divided the first number line (below) by eye into quarters and marked one half and three quarters. He then placed one-half on the number line below corresponding to its position on the first number line. He said that “three-fifths is smaller than three-quarters” and marked three-fifths to the right of one-half and to the left of three-quarters on the first number line.



Interviewer: “Where would one-fifth be?”

Student 4: “One-fifth is more than one-half, I think.”

Student 4 used a new number line and placed one-fifth to the right of one-half. The interviewer then asked where he thought one-third and one-quarter would be on the number line. The student then placed these two fractions in between one-half and one-fifth as shown in the diagram below. Despite his apparent correct thinking in the previous example, Student 4 unexpectedly lapsed into *larger-is-bigger thinking*.



Conclusion

Despite their procedural competence with fractions, some students in the early years of secondary school continue to exhibit whole number thinking about fractions. This whole number thinking can be expressed in several different forms. It often becomes evident when students are asked to decide which of two fractions is greater.

It is vitally important to ask students to represent their thinking on a number line. The teaching of checking strategies and asking students to represent fractions on a number line can assist some students to identify and correct their misconceptions. Unlike *default* whole number thinkers whose thinking seems to embody quite serious and deep seated misconceptions about fractions, the thinking of *residual* whole number thinkers may be more amenable to modification and self correction when they are presented with situations which draw attention to inconsistent and incorrect thinking. This study, however, did not set out to explore remedial strategies with the students interviewed.

In the early years of secondary school, the research reported in this study points to the importance of continuing to assist students to:

make multiple representations of fractions using discrete and continuous quantities
use a number line to represent accurately and to compare fractions
check results and to estimate answers
inter-relate the different procedures used
deal explicitly with instances of incorrect fractional thinking.

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