Primary School Children's Knowledge of Arithmetic Structure

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This paper is a report on the beginning of a much larger study that examines students' understanding and knowledge about the structures of arithmetic and generalising problems. Ninety-four students in their final years of primary school responded to a series of tasks designed to probe their knowledge of associativity and commutativity. The responses indicated that these students had failed to abstract from the experiences in arithmetic some of the mathematical structures believed necessary for the successful transition from arithmetic to algebra.

Introduction

Recent studies, such as Australian Year 12 students' performance in the Third International Mathematics and Science Study (1998), have reported the misconceptions many students hold not only with understanding the concept of a variable, but also in solving algebraic equations, and in translating word problems into algebraic symbols. While many students are experiencing difficulties, algebra still holds a pivotal role in the school curriculum in the 1990's (*Principles and Standards for School Mathematics*, Draft, 1998,). School algebra may be seen as a focus on understanding variables and their operations, and formulating and manipulating general statements about numbers (Kieran, 1996). The goals of algebra typically include: generalising arithmetic; studying the procedures used for solving certain kinds of problems; representing the relationships between quantities; and studying algebraic structures (Thorpe, 1989). Even with the rise of technology in the algebraic domain, students still need to have a fundamental understanding of the concept of a variable and basic algebraic structures (Kieran, 1996).

Recently in the literature the distinction has been made between algebra, algebraic thinking, and pre-algebraic thinking (Kieran, 1992). Beginning algebra students are required to move from arithmetic thinking to algebraic thinking. Pre-algebra is defined in the literature as the transition between arithmetic and algebra transition (Boulton-Lewis, Cooper, Atweh, Pillay, & Lewis, 1998). It is when students think about the numerical relations of a situation, discuss them explicitly in simple everyday language and eventually learn to represent them with letters (Herscovics & Linchevski, 1994). This transition is believed to involve a move from knowledge required to solve arithmetic equations (operating on or with numbers) to knowledge required to solve algebraic equations (operating on or with the unknown or variable). This transition is referred to as breaching the cognitive gap or didactic cut between arithmetic and algebra.

Two aspects are considered to be crucial in this transition stage. These are, first, the use of letters to represent numbers and, second, explicit awareness of the mathematical method that is being symbolised by the use of both numbers and letters. This involves a shift from purely numerical solutions to a consideration of method and process. Yet many students experience difficulties in achieving this transition (Boulton-Lewis, Cooper, Atweh, Pillay, & Lewis, 1998). Kieran and Chalouh (1992) suggest that most students are not given the opportunity to make explicit connections between arithmetic and algebra.

When operating in an algebraic world students are frequently required to manipulate symbols. This is often achieved by simply using transformational rules such as, "when you move something from one side of the equation to the other side change the sign". Students typically have little understanding of the relationship among quantities or the structural properties of the mathematical operations (Kieran, 1992). Typically these students cannot provide a rationale for the transformation. Kieran (1992) believes that a predominant reason given for this is that students have limited knowledge about mathematical structure.

An understanding of algebraic structure is typically derived from knowledge of the structure of arithmetic. In this instance, knowledge of mathematical structure is knowledge about the sets of mathematical objects, relationship between the objects and properties of these objects (Morris, 1999). The knowledge is considered to be about relationships between quantities (e.g, equivalence and inequality), properties of quantitative relationships (e.g., transitivity of equality), properties of operations (e.g., associativity and commutativity), and relationships between the operations (e.g., distributivity). In a beginning algebra course it is implicitly assumed that students are familiar with these concepts from their work with arithmetic. From repeated classroom experiences in arithmetic it is assumed that by inductive generalisation students arrive at an understanding of the structure of arithmetic. Thus, knowledge of structure is considered to be at a meta-level, derived from experiences in arithmetic. How do classroom experiences impact on students' ability to derive structure?

Previous research has documented ways in which students' arithmetic experiences constitute obstacles for the learning of algebra. Most of this research has focussed on the differences between the two systems, for example, differing syntax (Lodholz, 1993), closure (Kieran, 1992), use of letters as shorthand (Booth, 1989), manipulations (Booth, 1989), and equality (Wagner & Parker, 1993). Little research has focussed on students' ability to induce mathematical structure from their experiences in arithmetic.

The current Queensland primary curriculum number strand consists of 4 main components, namely, numeration, basic facts, an understanding of the four operations (addition, subtraction, multiplication, and division) and the algorithms (Department of Education, Queensland, 1987). The emphasis appears to be on arithmetic as a computational tool, that is, counting and calculation. Calculation typically involves the use of prescriptive algorithms (Department of Education, Mathematics Sourcebooks, 1988). Students' experience with number in the primary school appears to be limited to numeration, operations, and the implementation of the algorithms. We argue that the overriding emphasis on computational procedures at the expense of exploring relationships is largely responsible for children's limited understanding of arithmetic structure.

This study reports on the pilot stage of a larger study designed to investigate students' pre-algebraic thinking. In particular, it explores students' ability to understand mathematical structure and extract general relationships and principles from problem situations. The focus of this paper is on children's knowledge of the properties of the operations, that is, associativity and commutativity. The sample was drawn from the upper primary school. At this stage students have completed all formal experiences with the four operations.

The particular aims relating to this paper were to ascertain students' ability to

1. recognise the commutative and associative law;
2. explain these laws in everyday language; and
3. represent these laws in symbols.

Method

Instrument

A written test consisting of five tasks was developed. This paper reports on the results of the first two tasks. These two tasks focussed on identifying children's knowledge of associativity and commutativity in arithmetic. Each task consisted of four components. In the first segment students were given examples of the four operations and asked to indicate which were true or false (see Tables 1 and 5). The numbers chosen in the examples were believed to be small enough to allow students to complete the calculations without the assistance of a calculator. In the second segment, students were asked to create two more examples for the operations that they believed were true. They were then asked to explain the patterns they had discovered and asked to express the pattern using symbols such as ♥ and ∞ instead of using numbers. The instrument was administered under test conditions. Students were allowed ample time to complete each of the tasks and the test was calculator supported.

Participants

The test was administered to ninety-four children aged from 10 years to 12 years. Half the sample had just completed the second last year of primary school and the other half had just completed the final year of primary school, ready to make the transition to secondary school. The children attended three different primary schools in Brisbane. Each school was located in a middle to high socio-economic area. The participating schools were considered to be typical in their approach to teaching mathematics, that is, textbook supported and an emphasis on the algorithms.

Results and Discussion

Initially, the Year 6 and Year 7 students' results were analysed separately. χ^2 tests were used to identify the differences between the two groups. On the whole, the groups were not significantly different. Thus for reporting purposes, the two groups were combined (n = 94).

The first task related to gauging students' understanding of the commutative law. Students were initially asked to indicate for which operations the commutative law was true. These results are summarised in Table 1.

Table 1
Percentage Distribution of Responses for the Commutative Law Question (n = 94)

Operation	Example	True	False
Addition	$1 + 3 = 3 + 1$	94.7	5.3
Subtraction	$1 - 3 = 3 - 1$	11.7	88.3
Multiplication	$1 \times 3 = 3 \times 1$	95.7	4.3
Division	$1 \div 3 = 3 \div 1$	16.0	83.0

Most of the students recognised that the number sentences were valid for addition and multiplication. Of concern was the number of students who believed that the statements for subtraction and division sentences were true, especially as the students were in the final stages of their primary school experience.

Students were asked to create two more examples for the operations they believed were true. The results are summarised in Table 2.

Table 2
Percentage Distribution of Responses to "Create two more examples" (Commutative Law)

Operation	One example	Two examples	Three examples
Addition	67.0	21.3	1.1
Subtraction	0.0	0.0	0.0
Multiplication	60.6	13.8	1.1
Division	0.0	2.1	0.0

As can be seen in Table 2, there was a marked difference between being able to recognise that the commutative law held for the addition and multiplication examples and being able to create two more examples.

The next segment required students to explain in their own words the patterns they had discovered in the commutative law. Students' responses fell into six broad categories delineated from the data. The categories appeared to represent increasingly more sophisticated levels of response. Table 3 lists the categories and summarises the percentage of responses in each category.

Table 3
Responses to "Explain to a friend the patterns you have discovered" (Commutative Law)

Category	% frequency
1. No explanation	20.2
2. Offered a statement of the form "it goes in patterns"	46.8
3. Used only numbers in the explanation (e.g., 2 times 3 = 3 times 2)	13.8
4. Attempted to generalise but incorrect	2.2
5. Gave a valid explanation without the inclusion of the particular operation (e.g. turned around each time)	10.6
6. Gave a valid explanation with the inclusion of the particular operation (e.g., When you add and multiply it would be the same if you switch them around)	6.4

A typical category 6 response was, *the reason that 1 and 4 are true is because that when you multiply or add something then turn it around it will always equal the same thing, and when you subtract and divide the numbers and then turn them around it won't.* One student responded by focussing on the number sentences that were not true: *Because 1–3 will give you an answer of –2. While 3–1 will equal 2. And 1÷3 will give you an answer of 0.33 while 3÷1 will give you 3. While the other two operations are true.*

Two-thirds of the responses fell into the first two categories; that is, no response, or simply gave a statement such as "it goes up in patterns". Only 17 percent of the sample supplied a valid explanation. Six percent of the sample included the specific operations in their response.

This explain segment for the commutative law was the only component of the test where there was a significant difference between the Year 6 and Year 7 responses (χ^2 = 20.1, p = 0.003). A significantly greater number of Year 6 responses fell into the first two categories and no Year 6 student offered a category 6 response.

Table 4 summarises the percentage of students who were able to express their patterns in symbols.

Table 4
Percentage Distribution of Responses to Using Symbols to Represent the Commutative Law

Operation	Correct	Incorrect
Addition	67.0	3.2
Subtraction	0.0	2.1
Multiplication	45.7	2.1
Division	0.0	3.2

The students experienced greater difficulty in expressing the commutative law symbolically for multiplication in symbols than the addition law.

In summary, approximately 95 percent of students indicated that the examples for addition and multiplication were correct. Twenty one percent of students were able to create two more examples for addition and 13.8 percent were able to create two more examples for multiplication. Two thirds of the students were unable to explain the pattern in words, with the Year 6 students experiencing greater difficulty than the Year 7 students. Two thirds of the sample could express the addition pattern in symbols and under half could express the multiplication pattern in symbols.

The second task related to ascertaining students' understanding of the associative law. The format of the associative law task mirrored the format of the commutative law task (refer to Table 5). The results for each of the four components are summarised in Tables 5, 6, 7 & 8.

Table 5
Percentage Distribution of Responses to the Associative Law

Operation	True	False
(2 + 5) + 8 = 2 + (5 + 8)	88.3	11.7
(2 – 5) – 8 = 2 – (5 – 8)	12.8	87.2
(2 x 5) x 8 = 2 x (5 x 8)	89.4	10.6
(2 ÷ 5) ÷ 8 = 2 ÷ (5 ÷ 8)	19.1	79.8

Just less than 90 percent of the sample believed the addition and multiplication number sentences were true. As for the commutative law, of concern were the number of students who believed the number sentences were true for subtraction and division, with up to 20 percent of the sample indicating that the division number sentence was valid.

Table 6
Percentage Distribution of Responses to "Create two more examples"(Associative Law)

Operation	One example	Two examples	Three examples
Addition	54.3	19.1	0.0
Subtraction	1.1	0.0	0.0
Multiplication	50.0	12.8	0.0
Division	0.0	2.1	0.0

A small percentage of the sample could create two more examples for the addition pattern and up the multiplication pattern. Half the sample created only one more example for each. Up to a quarter of the sample could not create any more of the examples of the associative law.

Table 7 summarises the distribution of responses relating to asking children to explain the patterns they had discovered.

Table 7

Responses to "Explain to a friend the patterns you have discovered" (Associative Law)

	Category	% frequency
1.	No explanation	30.9
2.	Offered a statement of the form "it goes in patterns"	39.4
3.	Used only numbers in the explanation [e.g., (2+3)+4=2+(3+4)]	11.7
4.	Attempted to generalise but incorrect	2.1
5.	Gave a valid explanation without the inclusion of the particular operation (e.g. move the brackets)	13.8
6.	Gave a valid explanation with the inclusion of the particular operation (e.g. When you add and multiply you can move the brackets)	2.1

Compared with the commutative law, even more students experienced difficulties in explaining the patterns in language with 70 percent of the sample proffering no explanation or "goes up in patterns". Unlike the commutative law for this component of the task, there was no significant difference between the Year 6 and Year 7 students.

Table 8

Percentage distribution of responses to using symbols to represent the associative law.

Operation	Correct	Incorrect
Addition	51.1	2.1
Subtraction	0.0	2.1
Multiplication	42.6	5.3
Division	0.00	5.3

In summary, approximately 88 percent of students indicated that the examples for addition and multiplication were correct. Nineteen percent of students were able to create two more examples for addition and 12.8 percent were able to create two more examples for multiplication. Seventy percent of the students were unable to explain the pattern in words. Approximately half of the sample could express the addition pattern in symbols and under half could express the multiplication pattern in symbols.

In order to compare students' facility with the associative and commutative laws, each student was allocated a score for the first segment of each task accordingly to whether their response was correct or incorrect. The results of a paired t test indicated that there was no significant difference between students' facility to recognise the commutative law and the associative law ($t_{93}=1.704$, $p = 0.092$). There was also no significant difference between the level of response when students were asked to describe the pattern in their own words for the associative law and commutative law. In fact, most students performed poorly on both of these segments of the two tasks.

Implications and Conclusions

The results of this initial study highlight a number of concerns and implications for teaching and research in early algebra education.

There were a significant number of students who believed that the number sentences for subtraction and division were correct, with more students experiencing difficulties with division than with subtraction. This could occur for a number of reasons, for example, (i) a focus in early mathematics education on discovering relationships rather than also exploring non-relationships, (ii) little opportunities for students to explore their own conjectures and inductions, or (iii) teaching mathematics in a non-calculator supported environment making it difficult to explore the non relationships that exist in subtraction and division.

Many students experienced difficulties in finding more examples. Most could only find one more example, even though they were asked to find two. One reason for this could be that they made decisions in the first segment of the tasks purely on computational grounds without recognising the mathematical structure being represented by the number sentences. This could also reflect limited experience with the use of "=" as equality. Falkner, Levi and Carpenter (1999) reported that young children have enduring misconceptions about the equal sign. Even children as young as Grade 1 had already formed misconceptions. The reason for this, they suggest, is the equal sign is typically used in equations where one number comes after it.

The segment of the tasks relating to expressing patterns in everyday language was where almost all students experienced difficulties, with most proffering no answer or a basic response, such as, "it goes in patterns". The importance of natural language to the algebraic domain has been acknowledged in previous research ((Herscovics & Linchevski, 1994; Redden, 1996). Herscovics, Linchevski and Redden's research indicated that natural language description of number patterns seems to be a necessary prerequisite for representing the patterns in algebraic notation. While the present students were not strictly using algebraic notation to represent the patterns they had found, up to 67 percent of the sample could represent at least one of the patterns using symbolic notation. This raises a number of questions needing further research. How important is expressing patterns in everyday language for the successful transition from arithmetic to algebra? What is the relationship between language development and being able to describe mathematical relationships? MacGregor and Price (in press) indicate that this relationship is not straightforward. If expressing patterns in everyday language were important for the successful transition into algebra, what activities would assist students in this area of language development?

The results of this study point to a number of concerns with regard to the present curriculum. From this initial study, it seems that the majority of students are leaving primary schools with little awareness of the notion of mathematical structure. It seems that from the instances they are experiencing in arithmetic, they have failed to abstract the relationships and principles needed for algebra. More specifically, they failed to abstract the associative and commutative laws.

This has implications for the primary school curriculum. There needs to be more of a balance between calculations and searching for the implicit patterns in the operations. Students not only need many instances of relationships, they also need to explicitly discuss these relationships in everyday language. Present experiences do not appear to be reaching this balance. Students also need to explore non-relationships, such as $1 \div 3 = 3 \div 1$. They also need broader experiences in arithmetic encompassing activities where the equals symbol is used in equivalent situations (e.g., $2 + 3 = \square + 2$).

This paper reports on the beginning of a much larger study which includes interviews with selected students. These interviews will help clarify and illuminate many of the conjectures made from the written responses. Conjecturing from pen and paper responses about what children know or don't know is always fraught with difficulties.

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