Validation of the New Zealand Number Framework

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This paper reports the patterns of performance of almost 10,000 Year 4 to 6 students whose teachers participated in a project (ANP) designed to improve their professional knowledge, skills, and confidence. An important component of the project was the number framework, informed by work with *Count Me In Too*, and extended to cater for students at Year 4 and beyond. Data collected at the beginning and end of the project provided validation for the upper levels of the framework. A hierarchical sequence of development was evident, with additive part-whole thinking acquired before multiplicative part-whole thinking, which came before proportional reasoning. Support was also found for the idea that knowledge about numbers provides a necessary foundation for strategies.

It now seems to be generally accepted that national initiatives in mathematics education need to be based on sound research. In the UK, the Numeracy Taskforce claimed that its National Numeracy Strategy was research-based, and this claim was carefully scrutinised by Margaret Brown and colleagues (1998). In Australia, researchers involved in the development of numeracy initiatives have documented the extensive research base underpinning the initiatives. For example, Wright and Gould (2000) have provided a comprehensive account of the research behind *Count Me In Too* in NSW, while Clarke and his colleagues (2000, 2001) have done a similar exercise for the Early Numeracy Research Project (ENRP) in Victoria.

In 2000, New Zealand began developing a national number framework. The number framework was developed to give teachers a way of describing students' attainment on the basis of their number knowledge and problem-solving strategies. The framework has two main components: Strategy and Knowledge (see Ministry of Education 2001a, b). The Strategy component focuses on *how* students solve number problems, and the extent to which they use mental processes as part of their solution strategies. The Knowledge component encompasses key items of knowledge about the number system, including the identification and ordering of whole numbers and fractions, as well as grouping by tens. The two components are seen as interdependent, with Strategy creating new knowledge through use, and Knowledge providing the foundation upon which new strategies are built.

At the lower end of the framework are the counting stages, beginning with Emergent (a stage at which students are unable to count consistently), progressing through a series of increasingly sophisticated counting-all stages (One-to-one Counting, Counting from One on Materials, Counting from One by Imaging), to the highest counting strategy – Advanced Counting that incudes counting on and back, and skip counting. The lower end of the framework was informed by a pilot project with *Count Me In Too*, and is based on the work of researchers such as Les Steffe and Bob Wright (see Steffe & Cobb, 1988; Wright & Gould, 2000). The lower end of the framework focuses particularly on the transition from increasingly sophisticated counting strategies to using knowledge about the additive composition of numbers (ie, part-whole strategies). The first of the part-whole strategies, Early Additive Part-Whole, corresponds roughly to the Facile Number Sequence stage in *Count Me In Too,* and to the Derived Facts stage in the addition and subtraction framework

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of *Cognitively Guided Instruction* (Carpenter, et al., 1999). Part-whole strategies then become increasingly complex and more powerful as students progress from additive to multiplicative thinking, and then from multiplicative to proportional reasoning. The upper (part-whole) stages of the number framework were developed from the work of several internationally recognised mathematics educators (e.g., Behr et al., 1994; Confrey & Harel, 1994; Lamon, 1994; Pitkethly & Hunting, 1996), but unlike the lower (counting) stages, the upper (part-whole) stages have not been validated empirically on any large scale.

In 2001, the number framework became the basis for several initiatives, including professional development programmes for teachers at Years 1 to 3 (5-8 yr olds; Early Numeracy Project: ENP) and Years 4 to 6 (8-11 yr olds; Advanced Numeracy Project: ANP). Exploratory work with teachers at Years 7 to 10 (11-15 yr olds; Numeracy Exploratory Study: NESt) also began. The Advanced Numeracy Project provided an opportunity to examine the internal consistency of the framework, particularly at the upper levels. Data was gathered by teachers using individual diagnostic interviews with each child at the beginning and end of the project. Data gathered at the beginning of the project would have been affected little, if at all, by the teaching approaches taken in the project. On the other hand, teachers were inexperienced with using the framework and assessment tasks at this stage, and this may have affected the reliability of the data. Data gathered at the end of the project may have been more reliable than the initial data, but the teaching approaches taken in the project may have had some influence. For this reason, data from both the initial and final assessments was analysed to check for consistency.

This paper reports on the analysis of data gathered at the beginning and end of the project, and explores the interrelationships among different aspects of the framework.

Method

Participants

Just under ten thousand students (9964) from across New Zealand were assessed at the beginning of ANP. There were approximately equal numbers of girls and boys. The majority of students were of European descent (61.6%), approximately one fifth were Maori (19.5%), and the others consisted of Pacific Islands (10.2%), Asian (5.8%), and other ethnic groups (3.0%). Most of the students attended schools with catchment areas serving families of low (38.8%) or medium (40.3%) socio-economic status. Only one fifth (20.9%) of the participants came from schools in the high socio-economic decile band (deciles 8 to 10). Over the course of the project, just under a fifth of the initial sample (18.8%) were lost from the study.

Procedure

Each child was assessed using the Advanced Numeracy Project Assessment (ANPA) interview at the beginning and end of the project (see Ministry of Education 2001b). The questions and a brief description of the characteristics of each aspect of Strategy and Knowledge for levels 3 to 8 is presented in Appendix A and Appendix B. In assigning strategy stages, teachers focused on how the student solved the given problems rather than only attending to the correctness of answers.

Results

Particular subgroups of students were selected and their performance on the various Strategy and Knowledge components of the framework examined (see Table 1).

Table 1
Percentage of Children at Particular Levels on the Framework by Level and Component

		Strategies				**Knowledge**	
Level	**A/S**	**M/D**	**Ratio**	**Level**	**WNIDS**	**FRIDS**	**Group**
		Advanced Proportional Thinkers at the Start of ANP (n = 43)					
3				3		....	
4	2.3			4			4.7
5	9.3	2.3		5	4.7	16.3	11.6
6	88.4	23.3		6	95.3	30.2	18.6
7	....	74.4		7	....	14.0	23.3
8	....	....	100.0	8	....	39.5	41.9
		Advanced Proportional Thinkers at the End of ANP (n = 176)					
3				3		....	
4				4			1.1
5	4.0			5	1.7	4.5	2.3
6	96.0	8.5		6	98.3	12.5	12.5
7	....	91.5		7	....	21.0	19.9
8	....	....	100.0	8	....	61.9	64.2
		Advanced Multiplicative Thinkers at the Start of ANP (n = 424)					
3			0.2	3		....	0.7
4	1.2		4.0	4	.5	10.1	7.5
5	14.9		15.6	5	11.3	31.8	16.3
6	84.0		22.9	6	88.2	26.4	34.7
7	....	100.0	49.8	7	....	17.0	21.0
8	....	....	7.5	8	....	14.6	19.8
		Advanced Multiplicative Thinkers at the End of ANP (n = 1098)					
3			0.1	3		....	
4	.1		1.2	4		0.7	1.1
5	6.2		8.6	5	3.1	13.9	10.8
6	93.7		24.2	6	96.9	32.8	28.5
7	....	100.0	51.3	7	....	25.5	23.0
8	....	....	14.7	8	....	27.0	36.5
		Advanced Additive Thinkers at the Start of ANP (n = 1569)					
3		1.0	3.8	3		....	1.6
4		6.4	13.8	4	1.9	22.8	18.7
5		27.5	33.4	5	22.9	41.9	29.3
6	100.0	42.4	25.0	6	75.2	21.6	31.6
7	....	22.7	21.5	7	....	8.9	10.5
8	....	....	2.4	8	....	4.8	8.3
		Advanced Additive Thinkers at the End of ANP (n = 3004)					
3		0.1	0.8	3		....	0.1
4		2.5	7.4	4	0.3	2.7	5.3
5		20.4	28.6	5	12.1	31.8	23.8
6	100.0	42.7	30.8	6	87.6	36.2	36.7
7	....	34.3	26.9	7	....	17.4	16.4
8	....	....	5.6	8	....	11.9	17.8

The analysis began at the highest strategy level, focusing on those students who were classified by their teachers as Advanced Proportional thinkers, progressing though Multiplicative and Additive stages. The relationship of Knowledge to Strategy was explored by examining the performance of students who had low levels of Knowledge to see whether this limited the type of strategy that was available to the students.

Advanced Proportional Thinkers

Only a very small number of students at Years 4 to 6 were categorised as Advanced Proportional (0.4% initially, and 2.2% finally). If the Advanced Proportional stage builds on the Additive and Multiplicative stages, then the students at this top level should be proficient with addition/subtraction and multiplication/division. At both time points and for both aspects, most of the students were at the highest possible stage (see Table 1). A small number were operating just one stage below this. Measurement error may explain why one child was initially judged as two steps below the highest stage for Addition/Subtraction and for Multiplication/Division. The Advanced Proportional thinkers performed well on tasks designed to assess their Knowledge. All were judged to be at level 5 or above on Identification and Ordering of Whole Numbers and Fractions at both time points. All were at level 5 for Grouping, with the exception of two students (initially & finally) who needed to count by tens to 100 in order to find the number of tens.

Advanced Multiplicative Thinkers

A small number of students (4.3%) were classified initially as Advanced Multiplicative, but this had increased to 13.6% by the end of the project. Virtually all of the Advanced Multipliers were at the Advanced or Early stage for Addition/Subtraction (see Table 1). A small number (five initially, one finally) were classified as being at the Advanced Counting stage. The majority of the Advanced Multipliers were at level 5 or above on the various Knowledge domains. An exception to this was the 10.1% of students who could not identify unit fractions at the beginning of the project. However, knowledge of fractions was not necessary for the multiplication and division tasks the students were given.

Advanced Additive Thinkers

Only 15.7% of students were classified initially as Advanced Additive, but this had increased to more than a third (37.1%) by the end of the project. Virtually all of these students (98.1% initially, 99.7% finally) could identify and order numbers to 1000, and at least three quarters (75.2% initially, 87.6% finally) could do this with numbers to a million (see Table 1).

Students who use Counting Strategies for Addition/Subtraction

Table 2 shows the percentages of children who used counting strategies to solve addition and subtraction problems. Almost half (46.8%) of the students used counting strategies (either Count All or Count On) to solve addition problems initially. This had more than halved (20.8%) by the end of the project. The majority of these students used counting strategies to solve multiplication, division, and fractional problems. Of the students who used higher level strategies, virtually all used simple adding strategies (i.e., level 5).

Table 2
Percentage of Children who use Counting Strategies or can't Find Ten for Addition by Level and Component

	Strategies				Knowledge		
Level	**A/S**	**M/D**	**Ratio**	**Level**	**WNIDS**	**FRIDS**	**Group**
Children who Count All to solve Addition Problems at the Start of ANP (n = 698)							
3	100.0	75.8	76.4	3	21.8	....	44.0
4		21.1	18.8	4	54.9	86.7	50.7
5		3.0	4.7	5	20.8	12.6	4.7
6		0.1	0.1	6	2.6	0.7	0.6
7	....			7	....		
8	....	....		8	....		
Children who Count All to solve Addition Problems at the End of ANP (n = 131)							
3	100.0	73.3	64.9	3	19.8	....	39.7
4		24.3	27.5	4	56.5	62.6	51.9
5		2.3	7.6	5	23.7	34.4	8.4
6				6		3.1	
7	....			7	....		
8	....	....		8	....		
Children who Count On to solve Addition Problems at the Start of ANP (n = 3966)							
3		19.1	27.6	3	1.9	....	9.9
4	100.0	61.6	45.5	4	37.3	71.3	71.0
5		16.5	25.0	5	48.9	26.8	16.5
6		2.6	1.7	6	11.9	1.8	2.3
7	....	0.1	0.1	7	....	0.1	0.2
8	....	....	0.0	8	....	0.0	0.1
Children who Count On to solve Addition Problems at the End of ANP (n = 1550)							
3		11.8	18.2	3	1.0	....	3.0
4	100.0	64.3	49.3	4	30.3	35.6	63.9
5		21.7	30.3	5	54.8	57.4	28.5
6		2.1	2.0	6	13.9	6.2	4.1
7	....	0.7	0.3	7	....	0.7	0.4
8	....	....		8	....	0.7	0.1
Children who can't Find Tens in Numbers to 100 at the Start of ANP (n = 823)							
3	37.3	57.4	60.9	3	19.7	....	100.0
4	47.5	31.1	27.5	4	44.7	92.1	
5	12.2	9.0	10.1	5	28.6	7.3	
6	3.0	2.2	1.2	6	7.0	0.6	
7	....	0.4	0.4	7	....		
8	....	....		8	....		
Children who can't Find Tens in Numbers to 100 at the End of ANP (n = 113)							
3	46.0	60.2	57.5	3	24.8	....	100.0
4	40.7	36.3	31.0	4	50.4	72.6	
5	11.5	2.7	10.6	5	23.0	26.5	
6	1.8	0.9	0.0	6	1.8	0.9	
7	....		0.9	7	....		
8	....	....		8	....		

Students with Limited Knowledge about Numbers

An analysis was done of students who were unable to find tens in numbers to 100 (8.3% initially, 1.4% finally). Most (85% or more) relied on counting strategies to solve problems involving operations (see Table 2). Of the few who did use a part-whole strategy to solve these kinds of problems, virtually all were at the initial stage (level 5).

Discussion

The analysis gives strong support to the idea that the complexity of unit structures involved in number problems is the most significant indication of difficulty. Students appear to acquire strong control of additive unit structures before getting strong control of multiplicative structures then, in turn, proportional structures. The clarity of this pattern was surprising given the nature of the problems posed. In designing questions, the variables of number size and solution steps were constrained. This was done for two reasons. Firstly, students were asked to solve the problems mentally. Secondly, the problems were designed to be readily accessible to students who understood the unit structure. For example, to be classified as being at the Advanced Additive Part-Whole stage for multiplication and division, students needed to solve one of the following problems by deriving the answer from the given result:

17 x 6 = 102 so 18 x 6 = ?   27 x 2 = 54 so 27 x 4 = ?

In each case the strong control of additive structures was not a prohibitive factor in students solving the problems. 102 + 6 = 108, and, 54 + 54 = 108, are easy mental calculations. Yet only 2% of Advanced Counting (Counting on) students, and 22% of Early Additive Part-Whole (Derived addition facts) students could connect the given multiplication result with the unknown (final assessment). By contrast, 77% of Advanced Additive students could do so.

It appears that students who employ counting strategies on simple addition and subtraction problems apply the same strategies to multiplication, division, and fractional number problems. Early Additive Part-Whole students apply their preferred unit structures to multiplication, division, and fraction problems in the form of repeated addition strategies.

The findings provide substantial support for the hierarchical organisation of the Number Framework, with multiplicative thinking building on additive thinking, and proportional reasoning building on multiplicative thinking. The data also support the idea that having some minimal level of knowledge about numbers is a prerequisite for the development of part-whole strategies. Knowledge of the rudimentary place value conventions of "-teen", and "-ty" numbers (e.g., 10 + 4 = 14, 6 x 10 = 60) seems critical to the development of early additive part-whole strategies. Further trialing in 2002 will provide more data on the power of the framework hierarchy as a predictive instrument of students' strategies across a range of problems, and also show the validity of the framework as a pedagogical guide to teachers.

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References

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on teaching mathematics* (pp. 296-333) New York: Macmillan.

Brown, M., Askew, M. Baker, D. Denvir, H., & Millett, A. (1998). Is the National Numeracy Strategy research-based? *British Journal of Educational Studies*, *46*(4), 362-385.

Carpenter, T. P., Fennema, E, Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*, Portsmouth, NH.: Heineman.

Clarke, D., Sullivan, P., Cheeseman, J., & Clarke, B. (2000). The early numeracy research project: Developing a framework for describing early numeracy learning. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000:* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 180-187), Fremantle: MERGA.

Clarke, D. (2001). Understanding, assessing, and developing young children's mathematical thinking: Research as a powerful tool for professional growth. In J. Bobis, B. Perry & Mitchelmore (Eds), *Numeracy and beyond*: (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, pp. 9-26). Sydney: MERGA.

Confrey, J., & Harel, G. (1994). Introduction. In G. Harel & J. Confrey (Eds), *The development of multiplicative reasoning in the learning of mathematics.* (pp. vii-xxviii) Albany, NY: State University of New York Press.

Lamon, S. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds), *The development of multiplicative reasoning in the learning of mathematics.* (pp. 89-120) Albany, NY: State University of New York Press.

Ministry of Education (2001a). *Curriculum update #45: The numeracy story.* Wellington: Ministry of Education.

Ministry of Education, Curriculum Division (2001b). *Advanced numeracy project: Draft teachers' materials.* Wellington: Ministry of Education.

New South Wales Department of Education and Training (2001). *Count me in too professional development package.* Ryde, NSW: New South Wales: Department of Education and Training.

Pitkethly, A., & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics*, *30*, 5-38.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetic meanings and strategies.* New York: Springer-Verlag.

Wright, R., & Gould, P. (2000). Reviewing literature relevant to a systematic early numeracy initiative: Bases of CMIT. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 56-63). Fremantle: MERGA.

Appendix A

Questions & Criteria to be met at each level for selected components and Characteristics of the Framework

Ratio

8 *It takes 10 balls of wool to make 15 mittens. How many balls of wool does it take to make 6 mittens* (correct); or *There are 21 boys and 14 girls in Anna's class. What percentage of Ana's class are boys*? (correct)

7 *Of every 8 apples in the box, 3 are bad. There are 40 apples in the box. How many are bad*? (correct); or *There are 28 beans under the card. You need to take 3/4 of them. How many is that?* (solved using division & multiplication)

6 The 3/4 of 28 beans problem described above solved using addition and multiplication

5 *Here are 12 beans. You have to get one half of them. How many should you take?* (solved using addition facts)

Multiplication & Division

7 *There are 24 muffins in each basket. How many muffins are there altogether?* (solved using the distributive property or compensation); or *At the car factory, they need 4 wheels to make each car. How many cars can they make with 72 wheels?* (solved using the distributive property or compensation)

6 *17 x 6 = 102 so what does 18 x 6 equal*? or *27 x 2 = 54 so what does 27 x 4 equal?* (solved by deriving from known multiplication facts)

5 *Here is a field of cows. There are 5 cows in each row and there are 6 rows. How many cows are there in the field altogether? Here are 15 more cows. If they join the others in the rows of 5, how many rows will there be then? How many cows will that be altogether?* (solved using repeated addition)

Addition & Subtraction

6 *There are 53 people on the bus, and 26 people get off. How many people are left on the bus?* (solved using a part-whole strategy)

5 *On this page there are 15 birds and 27 cats. How many animals is that altogether?* or *I have 9 beans under this card and another 7 beans under here. How many is that altogether?* (2 bags of 10 beans added) *Now I have 29 beans under here and there are still 7 under here. How many is that altogether?* (solved using standard place value partitioning, adjusted doubles, or bridging through 10)

Grouping

8 *Here are some decimal numbers* (3.2, 1506.9). *How many tenths are in each number?*

7 *Suppose you had to make this much money* ($7 815, $253 000) *using only $100 notes. How many notes would you get?*

6 *Suppose you had to make this much money* ($60, $230, $4 520, $82 600) *using only $10 notes. How many notes would you get?*

Appendix B

Characteristics of each Aspect of the Framework for Levels 3 to 8

Strategies

Level	Addition/Subtraction (A/S)	Multiplication/Division (M/D)	Ratio
3	Counts all from one	Counts all from one	Uses equal sharing of objects
4	Counts On	Skip counts	Uses equal sharing of objects
5	Uses limited range of P/W strategies	Uses repeated addition	Uses addition facts
6	Uses full range of P/W strategies for add/sub'n	Derives from known multiplication facts	Uses multiplication & addition facts
7	……	Uses full range of P/W strategies for mult/div'n	Uses multiplication & division facts
8	……	……	Uses full range of P/W strategies for fractions

Knowledge

Level	ID & Order Whole Nos. (WNIDS)	ID & Order Fractions (FRIDS)	Grouping/Place Value (Group)
3	……	……	……
4	IDs & orders numbers to 100	……	Finds tens by skip counting
5	IDs & orders numbers to 1000	IDs unit fractions	Finds tens by using 10 tens = 100
6	IDs & orders numbers to a million	IDs decimals & orders unit fractions	Knows tens in any whole number
7	……	Orders decimals & nonunit fractions	Finds 10s & 100s, finds 10ths by ten 10ths = 1
8	……	Orders fractions, decimals & percentages	Finds 10ths in decimal numbers