Learning Through Interacting with Students' Ways of Thinking

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This paper reports on the ways in which one middle grade teacher listened to, supported, and learned from her students' mathematical reasoning within the context of a modelling task in data analysis. The teacher implemented a sequence of tasks that focused on the development of rating systems through selecting, ranking, and aggregating quantities. Analyses of the teacher's practices suggested that: (a) the teacher's mathematical background governed her understanding of both the modelling task and the students' reasoning; and (b) the teacher adopted new roles in her interactions with the students, including a focus on listening and observing, asking questions for understanding and clarification, pressing for explanations and justifications, and encouraging reflection on learning at a meta-level. This study illustrates the potential for using modelling tasks that provide a rich source for revealing not only the students' thinking but also the teachers' understanding of the students' thinking. Such tasks provide a learning tool for teachers as they listen to, interpret, and support students' emerging mathematical ideas.

The importance of listening and responding to students' ways of thinking about mathematical tasks has been emphasised in recent documents (e.g., Franke & Kazemi, 2001; NCTM, 2000). Within a given classroom, however, teachers are faced with the challenge of understanding the multiple ways that children might interpret a problem situation and the multiple paths they might take for refining and revising their ideas. By understanding how students approach a mathematical task and how their ideas develop would appear to provide a strong basis for teachers to interact with students in ways that would promote their learning (Doerr, submitted). While there has been considerable research on teachers' understandings of students' ways of thinking in elementary mathematics such as numeracy, geometry, and rational numbers (e.g., Fennema et al., 1996; Jacobsen & Lehrer, 2000), there has been limited research in the middle and secondary school years on significant topics, including data analysis.

In this research, we see ourselves as supporting teachers in constructing interpretations of teaching and learning situations that increasingly reveal the nature of students' thinking and the nature of the teacher's thinking about the student learning. To achieve this in the present study, we engaged seven middle grade teachers in teaching a sequence of modelling tasks involving the development of generalized rating systems for decision making. This sequence of tasks differed substantially along several dimensions from the usual tasks that the teachers used in their instructional practices. In particular, these tasks encouraged students to develop and explore significant mathematical ideas and to share and explain their understandings and mathematical reasoning with their peers and their teacher.

In this paper, we address the ways in which one of the middle grade teachers listened to and supported her students' mathematical reasoning during the first activity of the modelling sequence.

Listening to Students

Researchers have argued that teachers need to be more attentive to their students' mathematical reasoning, to interpret and analyse their students' mathematical ideas, and to challenge them to revise or extend those ideas in order to become more powerful mathematical thinkers (Crespo, 2000; Even, 1999; Reid, 2002; Schifter, 1998). Yet teachers do not automatically hear their students' reasoning or deeper understanding, rather, they tend to listen to their students' responses for the purpose of evaluating the correctness of their answers (Even & Tirosh, 1995; Heid, Blume, Zbiek, & Edwards, 1999). When using what Davis terms an *evaluative orientation* (Davis, 1996, 1997), teachers are primarily concerned with diagnosing and correcting students' misunderstandings.

In contrast to the evaluative orientation are the *interpretive* and *hermeneutic orientations* that teachers display towards listening to their students in the mathematics classroom (Davis, 1996). Teachers who display an interpretive orientation listen to their students' ideas with the aim of accessing their understandings; teachers with a hermeneutic orientation continually interact with their students, listening to their ideas and engaging with them in the negotiation of meaning and understanding. Drawing upon Davis' orientations, Crespo (2000) identified two major "interpretive turns" in pre-service teachers' interactions with their students (p. 178). The first turn was a change in the focus of the teachers' interpretation from correctness to meaning. The second turn involved a change in the interpretive approach itself; that is, the teachers moved from initial quick and conclusive judgments about the students' mathematical competencies to more reflective and provisional interpretive claims. However, evidence of hermeneutic listening was absent, suggesting that such an orientation to listening is not accessible to teachers unless some specific experiences in this approach are provided. This assumption is supported by the findings of Morgan and Watson (2002) who found that teachers tended to have difficulty in understanding and valuing student solutions that deviated from the response that the teacher was expecting.

When struggling with the difficulties inherent in listening to students' ways of thinking, teachers are faced with the challenge of responding in appropriate and effective ways to what they do see and interpret in students' activity. In responding to the multiplicity of conceptual developments that may be taking place in her classroom, the teacher needs to choose various strategies to further that development. Such strategies could include an understanding of appropriate representations and the connections among those representations, a repertoire of probing questions, or insightful ways of using computational technologies. In sum, effective teachers need: (1) an understanding of the multiple ways that students' thinking might develop; (2) ways of listening to that development; and (3) ways of responding with pedagogical strategies that will support that development (Doerr, submitted).

Tasks for Revealing Students' Thinking

To help teachers develop an understanding of students' thinking, we designed a learning experience that provided opportunities for students to reveal their ideas about developing a rating system (see below for a more complete description of the tasks as well as Doerr and English (2003)). A modelling perspective on task design leads to an instructional sequence of activities that begins by engaging students with non-routine problem situations that elicit the development of significant mathematical constructs and then extending, exploring and refining those constructs in other problem situations leading to a generalizable system (or model) that can be used in a range of contexts. Principles for designing such thought-revealing modelling problems have been put forward by Lesh and colleagues and are described elsewhere (e.g., Lesh, Hoover, Hole, Kelly, & Post, 1992). The students' descriptions, explanations, and justifications form an integral component of the models they produce. In contrast to many of the problem-solving situations students meet in school, modelling activities are inherently social experiences, where students work in small teams to develop a product that is explicitly sharable. Numerous questions, issues, conflicts, resolutions, and revisions arise as students develop, assess, and prepare to communicate their products.

In this study, however, our focus is on the teacher's understanding of the ways in which students might reason about the task and on how the students' ideas might develop. The ways in which the teacher understands the students' developing ideas will influence how she responds to their activities. Hence, this task for students provides us with a site for examining the nature of the knowledge of the teacher in the context of her practice.

Description of the Study

Setting and Participants

Seven middle-grade teachers and their students participated in this study. They were from a co-educational private school situated within a middle class neighbourhood in Australia. The teachers welcomed the opportunity to explore new ways to engage students in meaningful problem-solving activities. The teachers and the head of the mathematics department expressed their concerns that their students did not have enough opportunities to engage in mathematical problem solving and that many of their students were limited in their abilities to solve problems that they had not seen before. All the middle-grade mathematics teachers in the school, along with the head of the mathematics department and the school principal, were enthusiastic about participating in the project. Neither the students nor their teachers had experienced the type of modelling problems implemented in this study. In this paper, we focus on one of the teachers, Mrs L., who was a seventh-grade teacher with biology, not mathematics, as her primary teaching subject. This was her second year in teaching middle school mathematics and she expressed to us her lack of confidence in teaching mathematics and her eagerness to engage students in learning mathematics that was more investigative.

Procedures: Teacher Workshops

To support the teachers in developing their understandings of the mathematics of the tasks and of new pedagogical strategies to use with them, we held four meetings that were attended by all the teachers (except one who missed two meetings) and by the head of department. Since the teachers' current practice included only limited use of group work, pedagogical strategies for interacting with groups were used by the researchers in these meetings and explicitly discussed with the teachers. The meetings were intended to familiarize the teachers with the mathematics of the problem sequence by engaging them in a discussion of their own solutions to the problems and a discussion of anticipated student solutions. During the first meeting, the teachers solved the Sneakers Problem (described below) and compared their solutions. The researchers presented some student solutions for discussion, from their prior experience (Doerr & English, 2003) with the task. This was intended to illustrate some of the variation in student reasoning that could occur with this task. The researchers suggested to the teachers that they encourage and allow the students to develop, evaluate, revise and generalize their own solutions to the problems. In this way, the context of teaching the modelling tasks to the students was intended to be a site that would elicit and reveal the teachers' thinking about the mathematics of the task and about the pedagogical strategies to use with students who were engaged with the task.

Modelling Activities

The modelling sequence consisted of five problem situations that required students to create usable rating systems across a range of contexts (cf. Doerr & English, 2003, for detailed descriptions of the tasks). The core mathematical ideas were centred on notions of ranking, selecting and aggregating ranked quantities, and weighting ranks. For each problem, the students worked in small groups to analyse and transform a set of data for the purpose of making a decision. The sequence of tasks was designed so that the students could readily engage in meaningful ways with the problem situation and could create, use, and modify quantities (e.g., ranks) in ways that would make sense to them and in ways that could be shared, generalized, and re-used in new situations. Each of the teachers taught the complete modelling sequence over a period of 10 to 12 lessons, depending on the teacher. Our focus in this paper is on the teachers' interpretations of and responses to the students' approaches to the first problem of the sequence, namely, the Sneakers Problem.

In the Sneakers Problem, the students encounter the notion of multiple factors that could be used in developing a rating system for purchasing sneakers and the notion that not all factors are equally important to all people. The students were asked "What factors are important to you in buying a pair of sneakers?" This generated a list of factors where not all factors were equally important to the students; the students then worked in small groups to determine how to order these factors in deciding which pair of sneakers to purchase. The students naturally produced different lists. The teachers then posed the problem that the sneaker manufacturer needed a single set of factors that represented the view of the whole class; in other words, the group rankings needed to be aggregated into a single class ranking. The task elicited alternative ways that students might aggregate the ranks such as a frequency-based approach that counted the number of groups that assigned a given rank or a totalling approach that summed the values of the ranks given by each group. The context of teaching this model-eliciting task for students was intended to elicit the teacher's interpretations of and responses to the students' reasoning as the students engaged in making sense of the task.

Data Collection and Analysis

Of the seven teachers, we chose two for in-depth observation across the sequence of activities, based on grade level, on prior observations of their lessons, and on their mathematics background. For these two teachers, each lesson in the sequence was videotaped and audiotaped by the researchers and detailed field notes were taken. The videotaping focused on the teacher and her interactions and exchanges with the students in her class. Informal conversations with the teachers that occurred before and after class were recorded. The teacher meetings were audiotaped. The audiotapes from all lessons, teacher conversations, and teacher meetings were transcribed. The data analysis was completed in two stages. The first stage of analysis involved open-ended coding (Strauss & Corbin, 1998) of the field notes and the transcripts of each lesson. This was followed by viewing the videotapes for each lesson, and adding annotations and clarifications to the transcript that were visible from the videotape. Each author did this coding independently for the first lesson. We then met to compare our codes; differences in coding were resolved by finding references to early codes and making comparisons and revisions to the codes. We repeated this cycle of independent coding followed by comparison and revisions through three additional lessons. This led to a set of codes that reflected the revisions and refinements and that each researcher was able to use with a high degree of reliability between researchers.

The second stage of the analysis consisted of finding clusters of codes for each teacher that defined the critical features or characteristics for each lesson. These features describe the dominant events that governed the lessons, such as the ways in which the teacher encouraged student thinking, the ways in which she employed representations, and the incidents in which she asked for meaning, explanations, and justifications. In keeping with our theoretical framework, we examined how the teachers interpreted their students' responses and how they attempted to support the students' mathematical reasoning.

Results and Discussion

In this paper, we report on the interpretations and interactions of one of these teachers, Mrs. L, as she implemented the first modelling activity (The Sneakers Problem). We first consider how Mrs. L.'s own mathematical understanding developed as she interacted with her students and with the task. We then address the new roles that Mrs L. took on in her interactions with her students, and last, we illustrate how she encouraged her students to reflect at a meta-level on their learning.

Development of the Teacher's Mathematical Understandings

Mrs L. was often hesitant in her interpretation of the students' mathematical thinking and frequently checked her interpretations with the researchers. For example, on the first day of implementing the Sneakers Problem, Mrs. L. expressed concern when she felt her students were not "thinking mathematically." She repeatedly stressed to her students to use "numbers, not pictures" in their solutions to the problem. This emphasis on numbers during the early stages of problem implementation suggests that Mrs. L. saw a mathematical solution as having to have some visible signs of computation.

In expressing her concerns about the students' lack of "numbers," Mrs. L. asked the researchers how she should handle the situation:

> None of them [the students] have put numbers on it [the list]. I don't know whether just to <pause> if this is the list they've come up with, and we get them [the lists] back up [on the board] again, they're going to see that they still have very different lists because they haven't done anything mathematically. [Class transcript. Aside comment to researchers. Day 1. Lines 417-420.]

The teacher saw that many of the student groups were simply using their own opinions to aggregate the lists of factors, rather than combining the data from the lists in some mathematical way. The teacher recognized that this would generate another set of very different lists, which was the problem that they were attempting to solve. Mrs L. herself was uncertain as to how to proceed. We see her hesitancy as reflecting the limits of her understanding as to how student models might develop from their early non-mathematical attempts to more sophisticated strategies. This suggests, in part, that the teacher did not have a range of strategies in mind for how students might approach the problem, despite our initial teacher workshops. Her understanding of the mathematics was just beginning to emerge as the students engaged with the task. We also saw her hesitancy as reflective of a tension between her willingness to work with the researchers in not telling the students what solution strategy to use and her need for her students to develop solutions that were mathematically worthwhile.

Adoption of New Roles

As the students were engaged with the task, we observed Mrs L. begin to take on new roles in interacting with her students. These roles were characterized by (a) listening and observing and (b) asking questions for understanding and clarification. However, Mrs. L.'s view of the students' work was coloured by her desire to see evidence of a numerical process. For example, after she observed one group's approach, she engaged in the following exchange with them:

Mrs. L.: Okay, so what have you girls decided? Are you counting the number of times that it came up first?
Student: Yes.
Student: Majority.
Mrs. L.: Show that on here - like, if you were doing some maths, counting up how many, show it on this paper because you're going to tell the class how you came up with this list. So you can do all your working on that paper as well. We want to see what numbers you've used and how you've worked it out.
Student: Fashionable had like three votes so we're basically doing a majority vote.
Mrs. L.: Okay. Where could you show that?
Student: On the side.

[Class transcript. Day 1. Lines 378-388.]

At this stage in the task, Mrs. L appeared to be most concerned about whether or not the students had used numbers in their solution. Her comment to the students focused their attention on where they should show their computational work, rather than on what the work meant. In listening to the next two groups, Mrs. L. commented "Can we have some working on your sheet?" [Class transcript. Day 1. Line 398.] and then "I'd like to see some numbers." [Class transcript. Day 1. Line 410]. In an aside to the researcher, Mrs. L said "They're not really thinking mathematically.... None of them have [sic] put numbers on it." [Class transcript. Day 1. Aside to researcher. Lines 413, 417.]. As Mrs. L listened to the students, she initially focused on the use of numbers as presenting evidence of the students' mathematical thinking. She recognized that the students would need to do some computations in order to aggregate the lists of factors in a mathematical way. However, her comments to the students seem focused on the display of computation ("working on your sheet" and "to see some numbers") rather than on the actual strategies (or their meaning) that the students were using to aggregate the lists of factors. As the lesson developed, Mrs. L. continued to focus on the students' work with numbers; late in the lesson, Mrs. L. asked a group "What can you do with these numbers here?" and then "Anything else you could do with these numbers?" [Class transcript. Day 1. Lines 542, 546.]. However, there were also instances where Mrs. L. encouraged the students to explain their thinking so that she could understand the process that the students used with the numbers:

Mrs. L.: I'm not sure I totally understand how these [factors on the list] became 4th, 5th, 6th, 7th, 8th and 9th.
Student: Because more people thought the price should be 4th because if you have a look, 4th is price, 4th is quality, 4th is dah dah dah. But then three groups thought the price should come 4th.
Student: So three out of the six groups thought that.
Mrs. L.: All right. If only one out of the six thought comfort was third, why is comfort third?
Student: Because that has to be swapped with another one as well.

[Class transcript. Day 1. Lines 496- 502.]

In this instance, we see that Mrs. L. attempted to understand the process that the students used to come up with the numbers (in this case, the ranks for each of the factors). As she better understood what the students did, she then asked the students to justify their process ("why is comfort third?") as they explained how their rating system worked.

Reflection at a Meta-Level

Another way in which Mrs L. supported her students' development was to encourage them to reflect at a meta-level on their learning. Mrs. L. made a special point of encouraging her students to reflect on the opportunities for independent thinking provided by the modelling problem. At the end of the students' group reporting session, Mrs. L. asked her students: "Now what would we have expected, considering that our original six lists were very different from each other; what would we have expected to get from working on the ultimate list—this new list—what should we have seen?" In response to this question, the following discussion ensued:

Student: About five or six or all of the groups getting the same answer.
Mrs. L.: In what situation would we have all got the same list?
Student: If you'd told us what to find out.
Mrs. L.: Do you think I should have told you what to do?
Class: No.
Mrs. L.: Did you enjoy trying to think about it yourselves?
Class: Yes.
Mrs. L.: Did any of you think about what we've done over the last week or two in maths?
Class: Yes.
Mrs. L.: Did anybody try to apply some of the things that we've been learning about?. Okay, what area in maths have we been looking at?

[Class transcript. Day 2. Lines 584-594.]

This meta-level conversation reflected the changes in both Mrs. L.'s role and the students' role in thinking about the task: Mrs L. moved away from her former role of telling the students how to solve the problem to placing the onus on the students to think about the task themselves.

Concluding Points

Our study of Mrs L. has shown how the use of modelling tasks that foster different ways of thinking within a social context provide rich opportunities for teachers to learn about and learn from their students' emerging mathematical ideas. We saw how the mathematical background of Mrs L. influenced her understanding of the mathematics involved in the task and her responses to the students (cf. findings of Heid and colleagues, 1999). Over the course of task implementation, however, the teacher's initial, uncertain understanding of the underlying mathematical structure evolved and was accompanied by the teacher adopting new roles in her interactions with the students. She focused on observing and listening to the students, asking questions for clarification and understanding, and encouraging the students to describe, explain, and justify their thinking.

As we indicated earlier, recent research (e.g., Crespo, 2000; Davis, 1996 1997) has called for teachers to be more attentive and responsive to their students' mathematical reasoning, what Davis terms a hermeneutic orientation to their teaching. We believe we saw evidence of such an orientation in Mrs L.'s implementation of the modelling task. She valued the importance of carefully listening to her students' ways of thinking, and provided extended time for this to occur. In providing this time, Mrs L. shifted from an evaluative orientation in her usual classroom teaching, which we noted in our pre-study classroom observations, to an interpretive and hermeneutic one.

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