Coordination of Fractional Quantities: Cueing of Resources, Constraints, and Effect of Numeric Structure

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While previous work in the domain of proportional reasoning has primarily focused on the coordination of integer quantities, this study investigates how students coordinate fractional quantities. Fine-grained analysis of two seventh graders' responses to a set of systematically designed proportional tasks, shows how their knowledge of multiplication and division from the domain of integers affords them the necessary resources to coordinate the fractional quantities. Further, it succinctly shows how the numeric feature of the fractional quantities cues and constrains them.

Proportional reasoning has been shown to be influenced by a number of variables such as problem context, the numerical aspect of the problem parameters, type of ratios and order of missing value (Harel, et al., 1991; Kaput & West, 1994; Lamon, 2007). In his summary of research on multiplication and division, Greer (1992) highlighted the interfering effects of the numeric structure of problem parameters on problem conceptualization. For instance, in multiplication problems involving decimals, he pointed out that when the multiplier was changed from an integer to a decimal less than one, students divided rather than multiplied, a phenomenon he termed non-conservation of operation. Articulating a multiplicative relationship involving a decimal in contrast to an integer seems to require different cognitive demand. Along similar lines, observing the constraints that the participants in their study encountered in multiplicative problems involving decimals, Harel et al. (1994) raised the following issue: "the question of whether subjects encounter similar difficulties with multiplication and division problems that involve fractions has never been addressed" (p.381). This study attempts to investigate how the numeric structure, specifically fractional quantities, affords and constrains students in coordinating the two referent quantities in a specific proportional situation, at a fine-grained level of detail. Moreover, previous studies in the domain of proportion (e.g. Kaput & West, 1994; Weinberg, 2002) have primarily focused on the coordination of integer quantities. The present study looks explicitly at the coordination of two fractional quantities.

Mack (2001) showed how divisibility relationship between the parameters b and c in the fraction multiplication $a/b \times c/d$ facilitated the solution strategies of 6 fifth grade students. The most demanding situations involved multiplications where b and c were relatively prime, which constrained the students in their partitioning strategies. In other words, the ways in which students viewed and operated on units were influenced by the numeric features of the problem. Similar to Mack's study, the numeric features of the problem parameters are systematically varied here to show how particular resources are cued or constrained under given conditions. This study shows how two middle-school students coordinate fractional units at a fine-grained level of detail and the constraints that they encounter in the process, especially as the order of complexity of the units increases from unit fractions to composite unit fractions in the two referents. The two research questions for this study are: (1) How do the numeric features of fractional quantities enable or constrain the cueing of cognitive resources in the coordination of two quantities at a

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fine-grained level of detail? (2) What type of fallback mechanisms are deployed in the articulation of multiplicative relationships between two fractional quantities?

Analytical Framework: The concept of units

In his summary of research on multiplicative structures, Kieren (1994) underlines the role of units of quantity and their transformation "as a key and unifying activity in a child building a multiplicative structure" (p. 396). Behr et al. (1992) developed a symbolic system to represent units and to illustrate different ways in which they can be manipulated from a mathematical perspective. Units can assume different forms, at varying levels of complexity. Homogeneous units can be united to form two levels of units (or units of units) or three levels of units (or units of units of units) (Steffe, 1994). In turn, composite units can be decomposed to form lower level units. Level of units has to do with the different number of units that one can coordinate simultaneously at one moment. The construct of units is particularly relevant in the interpretation of students' reasoning with fractions. We use the fraction $6/4$ to illustrate the concept of two and three levels of units. The interpretation $6/4$ as 6 one-fourth units involve two levels of units, i.e., one-fourth units within 6 one-fourth units. If we are to produce one whole from $6/4$ then we have to take away $1/3$ of the 6 one-fourth units. Such a simultaneous reinterpretation of the one-fourth units in terms of one-third units involves the coordination of three levels of units: Six one-fourth units are interpreted as three composite units of two-fourths (Figure 1)

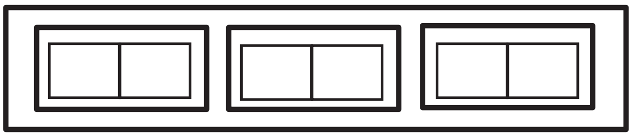

Figure 1. Three levels unit structure - One-fourth units within third units with six-fourth units.

Method

The study involved two seventh grade (12-year old) students. The students were interviewed on a range of multiplicative tasks in the domains of fraction, ratio and proportion. We can assert that both Aileen and Brian (pseudonyms) have a reasonably well-articulated number sense based on the seven weeks of interviews that the first author conducted with them. Evidence of their flexibility with multiplication and division can be inferred from their responses to Tasks 1-4 in Table 2. Compared to Aileen who tends to use paper and pen for her computations, Brian tends to do most of his calculations mentally, mentioning that he has a 'white-board' in his head.

A video camera was used to capture the moment-by-moment responses and to follow students' movements, gestures and facial expressions besides recording their spoken words and interjections. The responses were transcribed on a line-by-line basis. To further capture students' thought processes, a record of what they wrote or drew on the worksheets that were provided to them were kept. These inscriptions proved to be useful in supporting our claims. We used the rice-cooker problem as a context to formulate situations where respondents were required to coordinate two fractional quantities. Compared to Noelting's (1980) juice problems, the two quantities (rice and water) involved in this case are not mixed into one quantity. This enables students to differentiate clearly between the two quantities. Behr et al. (1992) used the term 'construct-the-unit' to refer to fractional problems in which students are required to construct the unit whole from a given fractional part in either discrete or continuous contexts. Similar to construct-the-unit problems in the domain of fractions, the rice-cooker problem involves proportional situations where the unit or whole of one of the two quantities is to be constructed (i.e., finding how much water is required to cook one cup of rice and how much rice can be cooked with one cup of water).

Table 1 shows the structure of the tasks posed in the two interviews as the numeric feature of the two quantities were systematically varied. Tasks 1 to 4 involve the coordination of two integer quantities. The aim behind the formulation of these four initial situations was to determine what the students can and cannot do in terms of the coordination of two integer quantities. The prime interest was in Tasks 6 to 9 that involve the coordination of two fractional quantities. The segment of data reported here comes from two clinical interviews that were conducted by the first author.

Table 1
Problem Structure

Tasks	Ratio of water to rice	Tasks	Ratio of water to rice
Task 1	$4 : 2$	Task 6	$3 : 2/5$
Task 2	$3 : 2$	Task 7	$2/3 : 1/2$
Task 3	$7 : 4$	Task 8	$3/4 : 2/5$
Task 4	$6 : 5$	Task 9	$1\,1/2 : 1\,1/4$
Task 5	$2 : 1/2$		

Each of the Tasks 1 -9 consisted of two parts, (a) and (b), as exemplified by Task 1. According to the rice-cooker usage guideline, we need 4 cups of water to cook 2 cups of rice. (a) How much water is required to cook one cup of rice? (b)How much rice can we cook with one cup of water? All the tasks were consistently worded and were posed sequentially as in Table 1. Note how the complexity of the units has been progressively increased in Task 2 ($3\,water \rightarrow 2\,rice$), Task 6 ($3\,water \rightarrow 2/5\,rice$) and Task 8 ($3/4\,water \rightarrow 2/5\,rice$).

Data Analysis

Coordination of two integer quantities

The first four situations involve the coordination of two integer quantities. The response of Aileen and Brian to the first two tasks (Tasks 1 and 2) showed that they could readily coordinate the two quantities, namely cups of water and cups of rice in situations with integer values involving divisibility relationships. This equally ascertained that they understood the problem situation. The next two questions in Task 3 and 4 asked them to coordinate two integer quantities which did not involve divisibility relationship. Table 2 summarizes the responses given to Tasks 3 and 4 involving the comparison of two integer quantities.

Table 2
Aileen's and Brian's strategies to compare two integer quantities

Task	Aileen	Brian
3(a)	A: "I was trying to get like one cup of rice by itself. So to do that you have to divide the four cups of rice by 4 and the 7 cups of water by 4. And so that would give you, one and three quarter cups of water for every one cup of rice."	He divided 7 by 4.
3(b)	She divided 4 by 7.	He divided 4 by 7.
4(a)	She started by comparing one cup of water to one cup of rice. A: "Because, you know that if it was like one cup for every. One cup of water for every one cup of rice, then we will need 5 cups of water. So if you go higher, if you did one and a half cups, that would be too much. So then I did, uh, I think I did 6 divided by 5 and that gave me one and one fifth.	He performed division. He deduced a 'method' as can be inferred: B: I think that I figure kind of method to do this. If it's how much water is required to cook one cup of rice then that just putting the water over the rice. So it's 6 over 5, which is one and one fifth. And if we are putting the rice, how much we can cook that in water (inaudible), that's putting the five over the six, which is five sixths.
4(b)	She performed an additive compensation rather than multiplicative compensation. A: So you would be able like. If you take one and one fifth cups of water to make one cup of rice. If you subtract a fifth from the water, then you subtract a fifth from the rice. So one cup of water would make four fifth cups of rice.	He divided 5 by 6. B: Same way as I did it last time. Just put the five over six instead of 6 over the 5 to give you five sixths.

In Task 4(a), we infer from Aileen's response that she started by setting a one-to-one correspondence between one cup of water and one cup of rice and then she incremented the ratio to $1 : 1\ 1/2$, and finally she divided 6 by 5. It appears that she also deduced the fraction $1/5$ from the integer 5 given in the problem. She checked her estimation by multiplying $1\ 1/5$ by 5 to get 6. On the other hand, Brian devised a method to solve the problem, dividing one quantity by the other, although he did not use that method in the later tasks. For the second part of the problem, i.e., how much rice can be cooked with one cup of water, Aileen used her initial deduction that $1\ 1/5$ cup of water cooks 1 cup of rice and subtracted one fifth from the rice to obtain four fifth, compensating additively rather than multiplicatively. The immediate question that crops up is why did Aileen change her strategy from division to first comparing one cup of water to one cup of rice and gradually incrementing the ratio? One hypothesis is that the closeness of the two given numbers, namely 5 and 6, prompted such an approach.

Coordination of an integer and a fractional quantity

As expected, Tasks 5(a) and 5(b) was readily answered by both students. In Task 6(a), the two participants used the building-up strategy (Kaput & West, 1994) to construct one unit of rice by coordinating $2/5 + 2/5 + 1/2(2/5)$ with $3 + 3 + 1/3\,(3)$ as can be inferred from the following response by Brian: "Because it's two fifth and then times two will make it four fifth and then half of that. You times that by two and a half, then it comes up to five fifth. You have to times this by two and a half too which gives you seven and a half." Such an additive strategy may have been favoured because of the relatively straightforward comparison between $2/5$ and one whole (one whole being $2\ 1/2$ times $2/5$).

The point to highlight from Aileen's response in Task 6(b) is that rather than producing 1 cup of water from 3 cups of water by dividing by 3, she first divided 3 by 2 (because of two fifths) to obtain $1\ 1/2$ cups of water and then attempted to reduce $1\ 1/2$ cups to one cup by correspondingly dividing $1/5$ by 2 (instead of 3) to obtain $1/10$ as can be deduced from her response: "Aileen: But one and a half cups of water equals one fifth; Interviewer: One and a half cups equal one fifth. Right; Aileen: And so if you go down to one it would be. Pause. I think it would be one tenth." She may have been prompted to divide by 2 because of the denominator of $1/2$. It should be highlighted that she used a similar strategy in Task 4 involving the comparison of two close numbers 6 and 5. On the other hand, Brian deduced that he should take $1/3$ of $2/5$.

Coordination of two fractional quantities (Tasks 7, 8, and 9)

Given that Task 7(a) involved the construction of one whole from half a unit, the two students readily answered the problem. For Task 7(b), Aileen added $1/3$ of a cup of water and correspondingly added $1/4$ cup of rice. Brian first determined that one cup of rice corresponded to $4/3$ cup of water. Then he subtracted $1/4$ of the $4/3$ from the water and $1/4$ from the rice to get $3/4$. Table 3 summarises the strategies used by Aileen and Brian to coordinate two fractional quantities in Tasks 8 and 9.

Discussion and Conclusion

This study investigated how the numeric features of fractional quantities enable or constrain the cueing of cognitive resources in the coordination of two quantities. The fine-grained analysis of the participants' responses to the rice-cooker problem illustrates how the coordination of two quantities is influenced by the type of numbers. Such numbers can cue or constrain students in deploying particular solution strategies. As the two quantities in the problems progressively changed from integers to fractions in Tasks 1 to 9, the two participants altered their strategies. In tasks involving integers, namely Tasks 1-4, the students almost readily deployed the division operation. However, this was not always the case when the problem parameters were fractions. For instance, the "method" that Brian derived in Task 4 in the case of integers, was not cued when fractional quantities were involved.

A striking example of the influence of the type of numbers in cueing particular solution strategies can be observed in Aileen's response to Task 8 ($3/4$ water → $2/5$ rice) which raises the following question: Why did she opt for the division strategy ($2/5 \div 3/4$) in part (b) rather than coordinating $3/4$ unit of water and $2/5$ unit of rice as she did in part (a) which involved the same multiplicative relationship? Given that the statements of parts (a) and (b) were exactly the same and the question for part (b) was posed just after part (a) in the same interview, it can be argued that it is the numeric factor that led her to change her solution path from coordinating the two quantities to using division computationally.

Table 3
Aileen's and Brian's strategies to coordinate two fractional quantities

Task	Aileen	Brian
8(a)	She coordinated $1/2$ of $2/5$ with $1/2$ of $3/4$ to deduce that $1/5$ cup of rice corresponds to $3/8$ cup of water. She the multiplied $3/8$ by 5 to get $15/8$.	Brian constructed one whole cup of rice from $\frac{2}{5}$ by adding $2/5 + 2/5 + 1/2(2/5)$ and coordinating it correspondingly with $3/4 + 3/4 + 1/2\ (3/4)$ or $7\frac{1}{2}/4$ cups of water.
8(b)	She divided $2/5$ by $3/4$ to obtain $8/15$.	Brian attempted to coordinate the two quantities to observe that $1/4$ unit of rice correspond to a certain number of 'thirds' which he could not determine. Finally, he converted the fractions to decimals and performed the division $0.75/0.4$ or $75/40$ rather than $0.4/0.75$.
9(a)	In her first attempt, she subtracted $1/4$ from each of the two quantities. Then, she divided $1\ 1/2$ by $1\ 1/4$ to get $6/5$.	He interpreted the $1\ 1/4$ cup of rice as 5 units of $1/4$ and attempted to subtract $1/5$ of $1\ 1/2$ from $1\ 1/2$ (which Aileen computed for him).
9(b)	She used two strategies: (i) She used the answer from part (a), i.e. one unit of rice corresponds to $6/5$ water to subtract $1/6$ from the rice and the water (ii) She divided $1\ 1/4$ by $1\ 1/2$ to get $5/6$.	Brian considered the 6 units of $1/4$ as three composite units of $2/4$ and interpreted the problem in terms of thirds. He subtracted $1/3$ of $6/4$ from $6/4$ (representing $1\ 1/2$ cup of water) and correspondingly subtracted $1/3$ of $5/4$ from $5/4$.

The constraining effect of fractional quantities can be observed by comparing the students' responses to Task 2 (involving the comparison of 3 units and 2 units) to that of Task 8 (involving the comparison of 3 [one-fourth units] and 2 [one fifth units]) . Although Brian could deduce that 1/4 unit of rice corresponds to a number of 'thirds' in Task 8, he could not determine what would that number be or what would be the numerator of those 'thirds'. As he could not make the multiplicative comparison using the fractional quantities in part (b), he shifted to decimals. This shift in representation of the quantities from fractions to decimals shows that the cueing of cognitive resources is also dependent on the numeric representation of the quantities.

Task 9 also demonstrates the effect of the numeric feature of two quantities in terms of cueing of resources. Aileen's interpretation of $1\ 1/2$ as $6/4$ facilitated the multiplicative comparison of $1\ 1/2$ and 5/4 as she interpreted 6 one-fourth units in terms of 5 one-fourth units. Such change of the fractional status from 3/2 to 6/4, i.e. the generation of congruent units (same-sized units, fourths in this case), allowed her to use her integer knowledge (of division) in the fractional situation. The ways in which fractional units are transformed to cue knowledge of multiplication and division from the domain of integers is regarded as a significant insight that the present study bring to the forth.

Reasoning with units

The interpretation of fractions in terms of their constituent units is a key resource that allows students to use their knowledge of multiplication and division from the domain of integers to multiplicatively compare two fractional quantities or to construct one unit of a quantity from the given multiplicative relationship. In Task 7, Aileen interpreted $2/3$ as 2 one-third units and this prompted her to use the halving operation to find one third of the given quantity and to ultimately construct 3 thirds. Similar observations could be made in Task 8, w here she interpreted $2/5$ as two one-fifth units. Analogously, Brian's interpretation of fractions in terms of units illustrates the flexibility that such a conceptualization affords in terms of the coordination of two fractional quantities. In Task 7, he interpreted $4/3$ $(2 \times 2/3)$ as 4 one-third units and constructed three thirds by subtracting one fourth of the four-thirds. In Task 8, Brian interpreted $2/5$ as two one-fifth units and $3/4$ as three one-fourth units and this allowed him to infer that "that two is three somehow". In Task 9, he interpreted $1\ 1/4$ as 5 units of $1/4$ and this allowed him to deduce that $4/5$ of 5 units of $1/4$ makes one whole. In the same situation, he interpreted $1\ 1/2$ as 6 units of $1/4$ and in turn reinterpreted these 6 units as 3 composite units of $2/4$ (3 levels of units). There was also a marked difference in the flexibility with which Aileen and Brian operated on the units. Aileen tend to work with two levels of units while Brian operated with 3 levels of units. Another ancillary resource (after the interpretation of the fractional quantities in terms of units) that could be observed at different junctures is the unit-rate proportionality schema as evidenced Aileen's and Brian's response to Task 8(a).

While analysing the data, we deduced two fallback mechanisms that were deployed in the articulation of multiplicative relationships between two fractional quantities. Firstly, Aileen performed an additive compensation (rather than a multiplicative adjustment) on two occasions, namely in Task 4 and Task 9 where she incorrectly subtracted $1/5$ and $1/2$ from $1\ 1/5$ and $1\ 1/2$ respectively to construct one unit of the required quantity. It is known that multiplicative structures (as is the object of the present study) often lead to the deployment of intuitive additive strategies. Kaput & West (1994) stated that sometimes the additive strategy appears as "a default procedure when the student is confused – a way to do something in the face of confusion" (p. 252). Besides the interference of intuitive additive strategies in the multiplicative situations, another fallback mechanism that could be observed was the use of estimation strategies. Such strategies are not always visible in the written work of students in actual classroom situations. In Task 4, which requires the coordination of 6 cups of water and 5 cups of rice, Aileen initially used a guess-and-check procedure to find which fraction fits the multiplicative relationship, $5x = 6$. As pointed out by Vergnaud (1988): "Some children, because mental inversion of the relationship $\times b$ into $/b$ is difficult, prefer to find x such that $x \times b = c$ (eventually by trial and error). This missing factor procedure, which is similar to missing addend procedures in subtraction problems, avoids the conceptual difficulty raised by inversion" (p. 131).

Implications

The case study presented here contributes to our growing understanding of the subtle strategies used and constraints that students encounter as they make sense of fractional quantities that are multiplicatively related. Of specific importance, are the ways in which the type of numbers conditions the cueing of the division operation. By concentrating on particular cognitive structures, the analysis presented in this two-student case study, is bound to be narrow in focus. Other situations may be examined to get more insight in terms of proportional reasoning involving fractions in future research endeavour. Also, the fine-grained analysis based on a m oment-by-moment decision making through the interviews reveals aspects of mathematical thinking that gives teachers much insight in understanding the ways in which students transform fractional units to be able to use their whole number knowledge to coordinate two multiplicatively-related quantities. Instruction on proportional reasoning should also lay emphasis on t he coordination of fractional quantities.

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