Does an Ability to Pattern Indicate That Our Thinking is Mathematical?

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Research affirms that pattern and structure underlie the development of a broad range of mathematical concepts. However, the concept of pattern also occurs in other fields. This theoretical paper explores pattern recognition, a neurological construct based on the work of Goldberg (2005), and pattern as defined in the field of mathematics to highlight what is intrinsically similar about the concept in these domains. An emerging model of patterning is proposed to describe this relationship.

In contemplating the term *pattern,* what comes to mind? A number pattern, lyrics in a song, patterns in a design, rhythms of nature, a sequence of events, chords in a tune, or a template to make an outfit? What about the use of the term pattern to describe how we learn from our experiences? The term pattern has distinct meanings across differing domains; what is intrinsically similar about the concept of patterning in each instance?

This paper will explore the concept of patterning across two domains, exploring the relationship between patterning as a neurological construct, the processes through which our understandings are encoded, and patterning as defined in the field of mathematics. Is there any similarity in the concept of patterning across these two domains? What is fundamentally different about patterning in each context, and is it viable to create a generalised model of patterning across these domains?

Theoretical background

Quasi-empiricism, a philosophical view of mathematics developed by Imre Lakatos (1976, 1978), recognises that “mathematical activity is human activity” (Lakatos, 1976, cited in Ernest, 1991, p. 37) and like all human endeavours is fallible and uncertain and therefore needs to be rigorously questioned (Ernest, 1991). If mathematics knowledge “is seen as connected with, and ... part of the whole fabric of human knowledge” (Ernest, 1991, p. 26), is there any difference with how we construct our knowledge generally?

There are four aspects of quasi-empiricism which this paper builds upon:

- Mathematics knowledge is fallible and like all knowledge can and needs to be questioned.
- Mathematical knowledge evolves; new mathematical knowledge is then part of an on-going process of knowledge creation.
- The primacy of informal mathematics: all formal mathematics is derived from informal human experiences.
- The genesis of mathematical knowledge: the creation of mathematical knowledge cannot be separated from the creation of human knowledge (Ernest, 1991, pp. 35-36).

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This paper asserts the view that mathematical knowledge is not created differently and should not be separated from human knowledge. In fact, this paper is essentially an inquiry into knowledge creation, identifying the possible mathematical elements involved in this process. I use the term *patterning* to describe the processes through which understandings are constructed.

The role of pattern has enjoyed a history of speculation in the field of mathematics. Steen in his groundbreaking text *On the shoulders of giants* (1990) has been widely cited as claiming that "mathematics is the science of patterns" (Steen, 1990, p. 1). Earlier, Piaget (1950) noted that "life itself is a creator of patterns" (cited in Lilejedahl, 2004, p. 24). Both are asserting the inherent role of pattern in the construction of knowledge, life knowledge and the formation of mathematical knowledge.

If "virtually all mathematics is based on pattern and structure" (Mulligan & Mitchelmore, 2009, p. 33), and "mathematical activity is human activity" which "produces mathematics" (Lakatos, 1976, cited in Ernest, 1991, p. 36), then is human activity also based upon pattern and structure? Are the elements of pattern and structure that are evident in mathematics also evident in human activity, and does this mean human activity is therefore mathematical? This question forms the basis of this paper. It is this relationship between mathematics and the nature of life itself that will be explored through the construct of patterning in both the mathematical and neurological domain. If this relationship can be substantiated, could a theory of mathematics evolve which views patterning, a mathematical construct, as the structure through which we create all our understandings?

Patterning as a neurological construct

In *The Wisdom Paradox* (2005), Elkhonon Goldberg, a world renowned neuropsychologist proposes that in building our understandings we are essentially patterning our experiences and retrieving them through the process of pattern recognition, which he refers to as an "ability to recognise a new problem as a member of an already familiar class of objects or problems" (p. 85). This pattern recognition eventuates from an accumulation of similar experiences, and "decision making takes the form of pattern recognition rather than problem solving" (p. 20). Goldberg cites the words of Herbert Simon (1966) in substantiating the role of pattern recognition as "the most powerful mechanism of human cognition" (p. 20). Goldberg refers to these patterns as "cognitive templates, each capturing the essence of a large number of pertinent experiences" which he relates to the acquisition of "wisdom … and a cognitive gain of aging" (pp. 21-22). These patterns enable us to rapidly recognise solutions to seemingly new situations as if they were familiar ones. The process involves being able to understand the elements involved, knowing what action to take and the possible outcomes that could result. Goldberg explicitly refers to this type of pattern recognition in terms of demonstrating the coveted attribute of wisdom and that these resilient patterns accumulate over a life time.

In everyday terms, Goldberg describes our ability to naturally sort and classify our experiences by relating what we know to past patterns of understanding; these being built through an accumulation of similar experiences over time. There are elements of new situations which have been experienced before and outcomes which have been previously tested. Goldberg claims that cognitive templates which encase these familiar elements are formed physically in the brain and are engaged through the process of pattern recognition.

The process of reconnecting to similar patterns over time creates neural structures, which Goldberg (2005) describes as "generic memories"; essentially these are "memories for patterns" (p. 125). "The more generic a pattern is … the vaster the set of experiences on whose overlap it emerged, the more robust … it is" (p. 125). Goldberg refers to these generic memories as an abstract representation of a set of similar experiences which is much more resilient than "concrete representations corresponding to unique things" (p. 125). Generic memories "capture the essence of a wide range of specific situations and the most effective actions associated with them," leading to high levels of competent, efficient decision making (p. 79).

Generic memory draws upon a network of common neural pathways related to the similar attribute of the experience they share. This overlapping of neural space becomes eventually "a shared network ... a mental representation of not any single thing or event, but rather the shared properties of a whole class of similar things or events" (p. 125), alluding to their inherent structure. The network not only represents the condensation of past experiences but also embodies information about "essential properties of class members" (p126), which can be added to, expanded upon and utilized in future situations, allowing rapid pattern recognition and application.

Goldberg acknowledges the role of language in "shaping our cognition by imposing certain patterns on the world" (p. 91); however he does not mention that pattern recognition, a neurological construct, could be mathematical. We have already explored though how this cognitive process involves sorting, classifying, identifying similarity, creating generic memories; all leading essentially to the development of abstract understandings and the application of these to predict in future situations. The purpose of this paper is to highlight how this everyday process is mathematical.

Pattern as a mathematical construct

Mathematics has been referred to as the "science of patterns … seeing and revealing hidden patterns are what mathematicians do best" (Steen, 1990, p. 1). In Steen's text *On the shoulders of giants: New approaches to numeracy* (1990), he suggests that the field of mathematics has significantly expanded because of this innate desire to "search for pattern" (p. 1). He elaborates on the significance of pattern in the field and work of mathematicians. "Mathematics seeks to understand every kind of pattern-patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns" (p. 8). In both instances Steen acknowledges the integral role of pattern in all aspects of mathematical study, and alludes to the role of patterns in life. He states how identifying and understanding the nature of patterns; and interestingly also the patterns in nature; is actually the process through which mathematical knowledge is created. This affirms the philosophical view that mathematical knowledge is derived from human knowledge and the desire to understand the world we live in.

In the field of mathematics, "a mathematical pattern may be described as any predictable regularity, usually involving numerical, spatial or logical relationships" and its structure is defined as "the way a pattern is organised" (Mulligan & Mitchelmore, 2009, p. 34). Replicating regularity involves recognising, predicting, and repeating what is deemed similar. Structural understanding emerges from generalising about the similarity; it involves exploring "a relationship that holds over the entire class of values, not only in isolated instances" (Papic, Mulligan, & Mitchelmore, 2011 p. 239).

Recent research (Arcavi, 2003; Lilejedahl, 2004; Mulligan & Mitchelmore, 2009; Papic et al., 2011; Warren, 2008) confirms that it is the actual process of exploring pattern and structure and developing visualisation that builds broader mathematical understandings. This process involves identifying patterns and similarities between patterns, constructing generalisations, the creation of abstract mathematical objects and structural awareness, as "abstracting patterns is the basis for structural knowledge, the goal of mathematics learning" (Warren, 2008, p. 759). In simplest terms, structural awareness is a recognition of structures within a pattern which are "the same … every pattern is a type of generalisation in that it involves a relationship that is 'everywhere the same'" (Papic et al., 2011 p. 240).

When a prediction is made that is based upon a generalisation about the pattern, this type of mathematical thinking leads to the ability to abstract (White & Mitchelmore, 2010). Structural understandings, an identified goal of mathematics, are developed through this ability to engage abstractly with patterns (Warren, 2008). The term *abstraction*, as used in the mathematical context, refers to "the degree to which a unit of knowledge (or a relationship) is tied to a specific context" (White & Mitchelmore, 2010, p. 1). There is a direct relationship between the degree of abstractness and the extent to which it is removed from specific situations, leading to a greater ability to generalise about "relevant conceptual attributes" across a range of contexts, so that "knowledge is more general and its applicability to different situations is increased" (White & Mitchelmore, 2010, p. 2).

Richard Skemp (1986), known for his pioneering work into the psychology of mathematics education, describes the process of abstraction as becoming "aware of similarities (in the everyday, not the mathematical sense) among our experiences", resulting "in some kind of lasting mental change" (Skemp, 1986, p. 21). Skemp explains that the act of naming objects is a form of classification. This involves identifying that a particular object belongs to a category based upon some predetermined criteria which is satisfied by the whole class of objects. Skemp links this process of classification to conceptual development, "a concept therefore requires for its formation a number of experiences which have something in common" (p. 21). White and Mitchelmore (2010) elaborate further on the nature of the similarities that Skemp is referring to, not in terms of superficial appearances but of underlying structure "in a sense the concept embodies or reifies the similarities" (p. 206). Essentially the concept represents the similarities and is the end product of abstraction.

The relationship between mathematical and neurological patterning

In this description of mathematical pattern, we hear familiar words that Goldberg echoed in his definition of pattern recognition—the ability to identify similarity, distinguish difference, and essentially to understand and apply the predictable elements to new situations in a generalised way. Mathematical abstraction resonates visibly with Goldberg's description of patterning as a neural construction of our generalised understandings, further affirming the theoretical perspective that mathematical knowledge cannot be separated from human knowledge (Ernest, 1991). It is this relationship between mathematical abstraction and pattern recognition which will be explored further.

White and Mitchelmore (2010) outline a theory of how students develop generalised mathematical understandings through abstraction and propose an approach to teaching, called Teaching for Abstraction, designed to support and strengthen this process. In this model, students engage in exploring a concept across a range of contexts and, as their sense of familiarity increases, learn to recognise similarity across contexts and develop generalised understandings, leading to a growing ability to predict and abstract.

The Teaching for Abstraction model has four phases, namely familiarity, similarity, reification, and application.

- In the familiarity phase, students explore a concept through engaging in a variety of contexts, becoming "familiar with the underlying structure of each context".
- The similarity phase involves frequent matching and explicit attention to structural similarities within the varying contexts and differences with other contexts.
- The reification phase moves students into operating with and developing abstract concepts.
- The application phase allows students to consolidate their understanding of the abstract concept through application to new situations (White & Mitchelmore, 2010, p. 5)

Similarly, Goldberg (2005) describes pattern recognition as an ability to pattern our experiences and draw upon these patterns in future instances. In doing so, we naturally go about seeking similarity, discriminating difference, and creating varying measures of understanding by relating new experiences to what is already known and understood. What is common about the experience helps to reinforce and expand the pattern even further. Over time these patterns are encoded as cognitive templates, predicting possible solutions to future problems becomes a matter of pattern recognition.

Each phase of the development of mathematical abstraction, as summarised in the Instruction for Abstraction model, can be aligned with Goldberg's description of pattern recognition, as shown in Table 1.

Goldberg describes how our everyday generalised understandings are constructed cognitively as generic memories and how in the long term this can lead to the rapid application of pattern recognition in new situations. This appears to be an intuitive process, as everyday understandings are constructed below our level of awareness. Situations can feel familiar because past experiences have merged to create a m ental construct, a patterning of similar understandings, which is challenged and changed with each new experience.

Table 1
Abstraction across domains[a]

Mathematical domain	Neurological domain
Familiarity: • explore a concept through a variety of contexts • become familiar with the underlying structure of each context	• Engagement with a range of situations set the scene for experiencing and sensing what is familiar about these situations.
Similarity: • frequent matching • explicit attention to similarities within and between varying contexts	• Further attention and engagement with familiar situations allows connections to emerge as we recognise what is similar about these experiences. • Engagement with these experiences enables similarity to be matched, measured, and understood. • Overlapping of neural networks encode the similarity experienced.
Reification: • moves students into operating with abstract concepts	• Over time a pattern is encoded as a generic memory, a mental representation of the similarities and shared properties of a 'type' of experience.
Application: • consolidation of the concept • application to new situations	• Pattern recognition refers to the ability to readily access this pattern in similar situations in the future.

[a]Information collated from White and Mitchelmore (2010) and Goldberg (2005), respectively.

In the mathematical domain the focus is on understanding and communicating the nature of patterns, which involves the ability to sort and classify; becoming familiar with the properties of mathematical objects across differing contexts; recognising similarities and using these to create, predict and generalise; leading to an ever increasing ability to deal with concepts in an abstract sense. Knowledge that is generalised across a range of contexts can be applied to new situations. Through this process of abstraction, freed from specific contexts. each generalisation "becomes a mathematical object in its own right" (White & Mitchelmore, 2010, p. 1). Increasing understanding leads to the development of "a point of view which guides our thinking" (Cassirer, 1923, cited in Van Oers, 2001, p. 284).

At each stage, the mathematical structure described through the Teaching for Abstraction Model (White & Mitchelmore, 2010) mimics the process of pattern recognition proposed by Goldberg (2005). There appears to be mathematics inherent in the way we construct generalised everyday understandings which implicitly guide our thinking and our future choices in new situations. Could the generic memories that Goldberg refers to in the process of pattern recognition be viewed as mathematical objects, a by-product of our ability to generalise?

In their analysis of the process of reification, Thompson and Sfard (1994) also elaborate on the nature of mathematical objects. "Objects ... are in a sense, figments of our mind. They help put structure and order into our experience (p. 11), "objectness comes from possessing coordinated schemes" (p. 16) that are linked together "because we feel somehow they represent the same thing" (p. 2). By this definition, coordination of thinking is aligned through the structural awareness we put in place to understand a concept. Structural awareness is developed through the process of generalising and abstracting the patterns we encounter. This concept ... this structure ... this pattern ... this object is re-engaged each time we encounter a similar experience, whether this experience is classified in mathematical or everyday terms.

In both the mathematical and neurological domain, structural awareness emerges from engaging with familiar patterns; similar elements of patterns across varying contexts merge to form structural understandings which encase the conceptual experience. This process leads to generalised understandings which can be applied in new situations to further engage and deepen the concept at hand. There are noticeable similarities in the terms used to describe the elements, processes and outcomes of patterning in both the mathematical and neurological domains.

Summary

Exploring the relationship between mathematical learning and the formation of everyday understandings through the construct of patterning has highlighted the similarities between the two contexts. In both cases, learning follows a similar cognitive process, involving the seeking of similarity, understanding of differences, and iteration of what is common through the repetition of experience. Patterning also explores the relationship between the elements, leading to a structural awareness of how the concept is organised and what is the same in each instance. This structural awareness enables one to generalise the relationships inherent in the pattern and apply an abstract understanding of the concept to make predictions in new situations.

Essentially we experience patterning through our everyday encounters and patterning is a process through which we construct our understandings. If mathematical activity though is human activity (Lakatos, 1976 cited in Ernest 1991) and all formal mathematics is derived from informal human experience (Ernest 1991), and if mathematical knowledge "is seen as connected with, and ... part of the whole fabric of human knowledge" (Ernest, 1991 p. 26), then can the mathematical concept of pattern be viewed as the formal embodiment of the informal sense of patterning we encounter through our life experiences?

It appears that the very processes through which we construct our understandings and ultimately experience life itself are essentially mathematical. Could a generalised model of patterning evolve, one which views pattern, a mathematical construct, as the structure through which we create and build our understandings? How could a generalised model of patterning support the development of both mathematical and everyday understandings? What implications could such a model have for our educational practices?

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