Primary Students' Understanding of Proof

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The notion of proof is arguably a fundamental concept in mathematics. Mathematics curricula expect students to develop understanding of proof through explaining and justifying their mathematical responses, and communicating these responses in coherent ways. This study reports the findings from a sample of students in Years 5 and 6 in two schools to a question that asked them to prove a mathematical statement of equality. Survey results from 56 students, and twelve follow-up interviews, were analysed using the SOLO model. Implications from the findings that most students could not use a zero statement to justify their responses, and that answers appeared to be related to language-based factors will be discussed.

Proof is one of the central aspects of mathematics, but there is, amongst mathematics educators, a broad view of what is meant by "proof". Milton and Reeves (2003) describe proof as a " … valid chain of reasoning" (p. 384). An acceptable proof, as indicated in curriculum documents, is determined by the context and the mathematical development of the learner. Students are expected to make, test and justify conjectures, explain solutions, and seek and evaluate alternative solutions (e.g., Australian Education Council, 1990; Department for Education and Employment, 1999). The National Council of Teachers of Mathematics (NCTM) *Principles and Standards* (2000) indicate that initially students will offer one example from personal experience as evidence of the truth of a statement. In later grades, students will offer several specific examples as proof of a proposition or conjecture and will accept that one counter example serves to disprove a statement (Carpenter & Franke, 2001; Carpenter & Levi, 2000). A more sophisticated approach relies on the use of generic examples that are representative of a class of numbers or objects, followed by the use of general rules and statements which disregard the need for substantiating illustrations.

In order for students to develop a useful concept of proof, they need to have experiences where proof, in some form, is required. Southwell (2002), in a useful review of the place of argumentation and proof in the primary classroom, indicates that processes of explanation and justification should be encouraged as precursors to formal notions of proof. Where students can explain and justify their answers to their own satisfaction this may be considered as a level of proof.

Milton and Reeves (2003) suggest that if proof is experienced in the number strand children's algebraic reasoning will develop. The move from arithmetic to algebraic thinking has been identified as a key aspect of mathematics curriculum reform (Carpenter & Levi, 2000). Arithmetic thinking consists largely of a procedural approach to executing operations on numbers in order to obtain an answer (Mason, 1996). In contrast, algebraic thinking focuses on the structure of a mathematical statement and uses such statements to describe generalisations succinctly and unambiguously (Macgregor, 1993). The shift in thinking requires, for example, that students be able to generalise arithmetic concepts, perceive relationships between numbers and operators (Carpenter & Franke, 2001), and accept "unclosed" answers (Biggs & Collis, 1982).

Many difficulties that students encounter when moving from arithmetic to algebra are well documented. In algebra, the arithmetic operators, add, subtract, multiply and divide, are used to relate terms rather than act as instructions to carry out a procedure, which is a common understanding of children (Esty & Teppo, 1996). Understanding of the role of the equals sign becomes a key determinant of progress, away from a perceived command to "get an answer" (Carpenter & Franke, 2001; Esty & Teppo, 1996) to one of relating mathematical statements. It is the understanding of the relational role of the equality sign that leads to the ability to generalise from arithmetic results (Carpenter & Levi, 2000; Macgregor & Stacey, 1999).

Another key aspect of algebraic thinking is the move to abstraction associated with the representation of a variable by a letter (Küchemann, 1981). Students may interpret letters as shorthand for the name of an object (e.g., m for matches). Others understand that letters represent numbers, but may interpret these numbers as alphabetical referents (e.g., $b = 2$ and $g = 7$) (Macgregor & Stacey, 1997). At a more sophisticated level, letters can be seen as representing numbers, but no particular numbers. Dawe (1993) refers to these as "dummy variables", or place holders, that is, the letters stand in place of any number and the equation becomes an identity: a statement of a general mathematical truth. Two groups of students who see letters as numbers can be identified. The first group of students recognises that numbers may be randomly assigned to letters, with the understanding that within an algebraic statement, a particular letter represents the same number. The second group of students sees letters as variables: that as the value of one letter is changed, so the values of others change according to the algebraic relationship connecting the letters (Küchemann, 1981).

The ability to recognise such relationships may grow with a developing language facility and the ability to understand the structure of everyday spoken and written text (MacGregor & Price, 1999). Students who can deal with everyday language by relating sequences of sentences or paragraphs to make meaning, and who recognise this process, can more effectively read mathematical statements because they seek out meaningful relationships expressed symbolically in those statements. MacGregor (1993) has noted that weaker readers read only from left to right, often decoding mathematical information in the order given. This "left-to-right" strategy may also reflect the arithmetic experience of students who are often presented with arithmetic problems to be completed in this order. The sequential processing of arithmetic problems and the resulting procedural understandings may be countered by students meeting arithmetic statements which are not closed, or have the equality sign placed so as not to lead only to left-to-right processing (Carpenter & Levi, 2000). Subsequent difficulties that ensue when students encounter algebraic statements may be diminished by such approaches (Carpenter & Franke, 2001), although Carpenter and Levi (2000) note the persistence of students' understanding of the equality sign as an operator.

Research questions

This report describes the findings from a study in which students from two Year 5/6 composite classes were presented with a generalised statement of equality and asked to show that the statement was true. The research questions for the study were:

1. How do upper primary students "prove" a straightforward statement of equality presented in abstract terms? and

2. To what extent do language factors, such as left-to-right processing impact on students' algebraic thinking?

Methodology

The 56 students involved in this study were in two "Opportunity Classes" (OC) in two different primary schools in a NSW rural town. To be included in these classes, students undertook a selective test in Year 4 and, on the basis of the ranked results, the top students were invited to join one of the two OC groups. They remained in the group for two years, throughout Years 5 and 6. For the purposes of the study reported here, no distinction is made between the responses of Year 5 and Year 6 students, since they were in classrooms where they experienced the same teaching program.

As part of a wider study of the mathematical proficiency of students in OC groups, students attempted the following question, adapted from one published by Reys and associates (2001):

The teacher writes this equation on the board. How would you show it to be true?

$$g + b - b = g$$

The question was answered twice, in June and November, as part of a larger written survey, intended to provide two measures of proficiency at different points in time. The first survey was presented by the class teachers, using an administration manual prepared by the researchers. The second was administered by researchers. Both surveys required students to provide an individual written response. On the same day as the second administration, six students from each class, selected by the teachers as being poor, average or good at mathematics, were interviewed by researchers. The algebra question was also presented in the interview, and the students were shown their written responses in order to stimulate their recall of their thinking. Extension questions were also asked, with a focus on the order of the letters and operators in the equation.

Responses were coded using the SOLO (Structure of the Observed Learning Outcome) model of Biggs and Collis (1982, 1991) to judge their quality. The basis for this judgment was the structural complexity of the performance, described as Unistructural (U) – making use of one aspect only of the information, Multistructural (M) – using several pieces of information, usually in sequence, and Relational (R) – providing a response that ties together all aspects of the information supplied to give an integrated solution. This "U-M-R" cycle has been extensively documented in school education, and two sequential U-M-R cycles have been identified in many different situations (Pegg, 2003). Although, U-M-R cycles may occur in any of the modes of thinking identified by neo-Piagetian researchers (Biggs & Collis, 1991), in this study, the target mode of thinking was the concrete-symbolic mode.

Results

Survey Responses

In the first survey, 17 students did not answer the question. The remaining 39 responses were coded using SOLO. Results from the first survey are summarised in Table 1, including examples of the responses to exemplify the coding. Responses from students who showed little understanding were categorised as prestructural (P); those providing a single numerical example were classified as unistructural (U_1) in the first cycle in the concrete-symbolic mode; multistructural (M_1) responses in the first cycle in the concrete-symbolic mode were characterised by substitution of numbers for letters; and first cycle relational responses (R_1) were those which acknowledged some abstraction but provided a concrete, numerical example. Two students gave responses that recognised the equality relationship $b - b = 0$, and that did not require a numerical example. These were coded as U_2, a second cycle unistructural response, because they implicitly understood the relationship inherent in the statement but were able to focus only on the notion that because b was both added and taken away the b terms cancelled out and left g, making the statement correct.

Table 1.
Survey 1 response codings

Code	Number	Example
P	3	*g* + = unknown-*b* *g* + *g* not enough info
U_1	8	Because *g* could equal 5 and ***b*** could equal 1.
M_1	20	The way I would answer this problem is by putting numbers in the place of letters, I would also split the problem into different parts. The answer is True 3 + 2 – 2 + 3 3 + 2 = 5 5 – 2 = 3
R_1	6	There is a *g* you add a *b* and take the *b* back off which takes you back to *g*. 6 + 4 – 4 = 6 is the same as $g + b - b = g$
U_2	2	*g + b – b = g you cross out the b's because they equal each other out, so g = g*

About half of the first survey responses (51.3%) were coded as M_1. This was not unexpected, given the age and mathematical experience of these students. The class teachers indicated that they had completed little work in the patterns and algebra strand of the syllabus at the time of the first survey.

Of the 56 students in the study, 33 provided valid responses to both surveys. Figure 2 shows the proportions of responses in each SOLO level in each survey from these matched students. The results indicate that there was some growth in understanding over the period of the study. The higher proportion of R_1 responses suggests that students are beginning to argue in terms of generalisations, although still with the need for a concrete example.

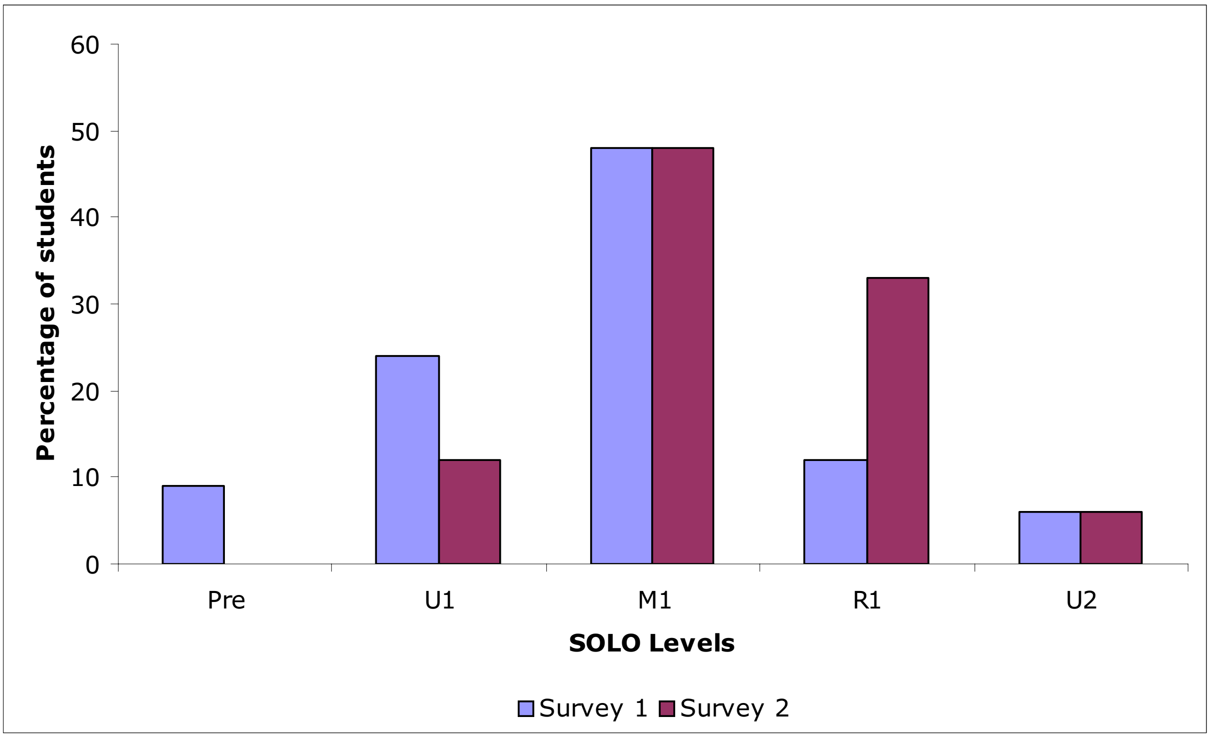

Figure 1. SOLO categories from matched students' survey responses.

Interview Responses

Interviews allowed a deeper exploration of the students' understanding. They were initially asked the same question, and, after they had responded again, were prompted with their previous survey answer and asked to explain what they had done. Of those students who changed letters for numbers, all chose small numbers at random: " … the first ones that came into my head" (TP023). No student in the interview demonstrated use of an alphabetic referent, although three survey responses indicated that they had used $g = 7$ and $b = 2$.

Three students in interview explicitly recognised that $b - b$, or a numerical example, was zero. Others indicated that $b - b$ made no difference to the equation, implicitly recognising the equality relationship: " … you just add b and you minus it straightaway so it doesn't really matter" (TP023). Several other students in interview gave a similar response, stating that $b - b$ cancelled out. These students demonstrated an implicit understanding of the zero relationship, but through drawing on their understanding of arithmetic operations rather than recognising the inherent relationship. These students used language such as "It's just a matter of plussing and minusing it" (TP025). One student demonstrated this kind of understanding, although using incorrect algebraic notation: "You add something and then you take it away. $g + b$ can be gb then if you minus it, it becomes g" (TP006). The same student's survey responses showed an R1 response on both occasions:

> This is true because: $g = g + b - b$ b is the same number as g $g = g$ because the b and the b are like non-signifigant [sic]; you add it on, and then you take it away again, so g would stay the same as g. (TP006, Survey 1)

If *b* has input into *g*, then the answer will become different, but then if *b* is out putted, then it is back to normal, so $g + b = gb$ $gb - b = g$. eg. if you wrap up a present, it becomes different, because before you could see what it was, but now you can't, and then when you unwrap it, you can see it, so it is back to what it originally was. (TP006, Survey 2)

In the first survey, despite some confusion shown by the statement " ... b *is the same number as* g*...", the student demonstrated emerging understanding of the relationship. The second survey response, although coded at the same level, seems to indicate that the student is struggling to reach a more complex understanding, with the use of words such as input and output. Nevertheless, a concrete example is still needed, albeit a less directly related one. The strong visual image of the present suggested that the student used some visual imagery in her reasoning. This inference was reinforced when the student was asked to rewrite the equation so that it would still be true. She substituted triangles and squares for* g *and* b*. When prompted to use letters, she replied that different letters would " ... be like a different picture".*

When the equation was presented to the students rewritten as $b + g - b = g$, most students checked by substituting numbers, carrying out the arithmetic operations from left to right. Providing the arithmetic was correct, they were assured of the truth of the statement. When the equation was rearranged as $-b + g + b = g$, similar results were seen. A common response was to substitute numbers and compute from left to right in sequence: $-b + g$ and then add *b*. Once again, if the arithmetic was right, students felt the equation to be true.

Student TP023, who used arithmetic language in interview, was one who showed a higher level response in the second survey. The initial survey response was "Say if *g* was 8 and *b* was 4. It would be $8 + 4 - 4 = 8$. Obviously this is correct." This response was coded as M_1 because, although it gave a single example, the inclusion of "say" qualified the response and indicated an implicit understanding that any number could be used. In the second survey the student wrote

For eg: $g = 5$ $b= 14$ $5+14 - 14 = 5$ Whatever the start letter or number in this case is, it will end up the same number!! $b - b + g = g$ $-3 + 4 + 3 = 4$ TP023

This was coded as R_1, because there was an explicit stated understanding of the relationship, although the concrete example was still necessary. It was also noted that the numerical example given in the second survey response showed a rearrangement of the terms, showing – 3 first. In interview, however, the student did not appear to acknowledge the rearrangement other than in arithmetic terms.

When presented with $-b + g + b = g$, two students were particularly uncomfortable with an equation beginning with a negative number, as shown by the following exchange where S indicates the student and I is the interviewer:

S. You usually start with a plus. If it's minus b then it is nothing and then plus g is g and then plus b which means it has to be more than g unless it goes down into decimals but I don't think it would.

I. So what is the problem with this equation [– b + g + b = g] ?

S. You are taking it away before you plus it because it is minus b first. (TP025)

This student was convinced that the equation was true when she substituted numbers, but could only proceed arithmetically left to right. She could not provide an alternative explanation as to why the equation was true. One other student could not accept an equation starting with a negative number:

Does it start with zero? (TP014)

When the interviewer indicated no, the student proceeded:

> Say it started on eight, then b equals seven. So it will be one plus g equals nine. Eight minus seven equals one, plus nine equals two, plus seven equals nine. (TP014)

Not only is the logic very confused, but the student also needed to invent a positive number with which to start the equation.

Another student responded to the $-b + g + b = g$ equation by reorganising the letters when the interviewer asked:

> I. Is there anything special that you notice?
>
> S. Not really. Just that it goes b, g, b, g.
>
> I. Just like your minus one, two, one, two?
>
> S. You could probably go [writes] – g + b + g = b (TS009)

Many students rearranged the letters, swapping the b for g, when asked to rewrite the original equation ($g + b - b = g$) in a different way so that it would still be true, At first sight, this facility suggests that the students understood the underlying relationship. Not until they were confronted with $-b + g + b = g$ did it become apparent that they were in fact reading the equation from left to right, and not seeing the mathematical relationship between b and $-b$.

Discussion

The algebraic thinking and understanding of proof demonstrated by the students in this study was consistent with that reported by other researchers. A majority of the responses, in both surveys and interviews, offered numerical examples as proof of the statements, in keeping with the findings of Carpenter & Levi (2000). Some students did appear to be moving towards a more abstract understanding, but these students were in a minority, as might be expected from students in the upper years of primary school, although there was some growth in understanding over the period of the study.

These students appeared comfortable with substituting numbers for letters, possibly reflecting common puzzle type activities often experienced in primary schools. They were not able, however, in most instances, to explicitly show understanding of the relationship that was expressed by the equation $g + b - b = g$. When they offered numerical justification, all executed this sequentially from left to right. If they substituted numbers, most students could articulate in some way that $b - b$ was zero, and that $-b + b$ cancelled in the context of $g - b + b = g$. Only two students in interview could see the relationship that was the key to the equation when they were presented with $-b + b$. It seems that most students were not seeing the equation as a whole, but were reading it as a set of executable instructions.

Of interest also was the apparent lack of understanding of the role of the equals sign. No student rearranged the equation in such a way that the position of the equals sign changed relative to the letters or numbers used. This is consistent with arithmetic understanding of equals as a command (Esty & Teppo, 1996), and may reflect a lack of experience with different arrangements of arithmetic statements.

The influence of language was seen in some responses. In particular the high incidence of left-to-right processing was noticeable, and was more evident when students were presented with the rearranged equation $-b + g + b = g$. However, the response of student TP023 in the second survey indicated that some students at least were able to consider the equation wholistically, even though this same student could not articulate this level of sophistication in interview. Teachers need to encourage students to read and write arithmetic statements of equality in various forms in order to develop an acceptance of unclosed statements.

The students in this study were able to satisfy themselves of the truth of the equation $g + b - b = g$, mainly through numerical substitution. Although still a long way from producing a rigorous proof, it seems that students at upper primary level can demonstrate convincing arguments concerning statements of equality, even when these statements are presented in abstract terms.

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