

Teacher Approaches to Graphing a Difficult Cubic Function

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Responses from 42 attendees at a mathematics teachers' conference to a task involving the search for a complete graph of a difficult cubic function were analysed. Neither a specific initial action nor the application of mathematical knowledge guaranteed an immediate global view. However, flexibility of approach, use of the automatic range scaling features, positioning the view shown by the calculator with one's mental image of the function, and the application of graphing calculator knowledge in conjunction with mathematical knowledge facilitated the teachers' solutions. Focussed experiences involving these should provide the basis of lessons designed to maximise student learning in this topic.

Doerr and Lesh (2003), in a recent discussion of "improving the practices of teaching and learning" (p. 262), suggest "progress towards changes in teaching practices remains slow" (p. 262). In the same vein, Barton (2003) admonishes, it is "about time that we ... started looking at the effect of technology on learning" (p. 85). He calls for "much more work in the area of *what* use can or should be made of the technology, *what* effect this has, and *how* it can best be brought about" [italics in original] (p. 85). A challenge exists to "capture and interpret characteristic forms of technology use exhibited by students and teachers" (Galbraith & Goos, 2003, p. 364). In previous work (e.g., Brown, 2003b) student approaches to graphing a difficult cubic function have been presented. However, it is often difficult to ascertain what the practice of teachers actually is. In this paper, approaches by mathematics teachers to the same task, but in a different "situation context" (Wedegge, 1999, p. 207), are presented, analysed, and discussed in order to begin to address some of the areas identified by Barton above.

Rationale

The approaches mathematics teachers use in engaging with a particular task in a graphing calculator environment provide some insight into the practice of these teachers in their own classrooms. Watson (2003), in redefining the role of the teacher, suggested one responsibility was "to narrow the range of what it was possible to learn [from a lesson] and thus increase the opportunity to learn appropriate mathematics for as many students as possible, while making sure that they all had access to the patterns under consideration" (pp. 36-37). Some, such as Akhras and Self (2002), may see such an approach as a negative affordance of classroom teaching but alternatively, it can be thought of, possibly more fruitfully, as a constraint in order to learn. To be able to follow Watson's advice in any situation we need to be able to capture the collective wisdom of teachers who have taught a specific topic to numerous students in many different circumstances. Whilst we cannot infer teaching approaches from one teacher's approach to a task, the analysis of a collection of approaches may provide insight as to the lesson elements required to maximise students' opportunities for learning that which we determine is essential for a particular topic. It may then be possible to achieve this narrowing of possible lesson outcomes referred to by Watson (2003).

Methodology

Participants and Data Collection

The participants were a selection of the attendees at the 2002 annual Mathematical Association of Victoria conference. Participants attended one of two sessions for Year 10-12 teachers presented by the author where she reported preliminary findings of her research involving senior secondary students and their solutions to a particular problem task (Brown, 2002). Participants attempted the same task and recorded their (beginning) solution on a record sheet (see Figure 1). At the end of the presentations participants were invited to submit the record of their solution. A self-selected group, that is a voluntary response sample (Moore & McCabe, 1998, p. 257), did so and these data are analysed and reported here. Two sessions were presented with 25 (69%) and 17 (68%) participants respectively submitting solutions. Whilst this convenience sample is not representative of the broader population of mathematics teachers, the data are not being used for predictions about this population; rather, this exploratory research was undertaken to increase understanding of the possible scope of approaches undertaken by teachers.

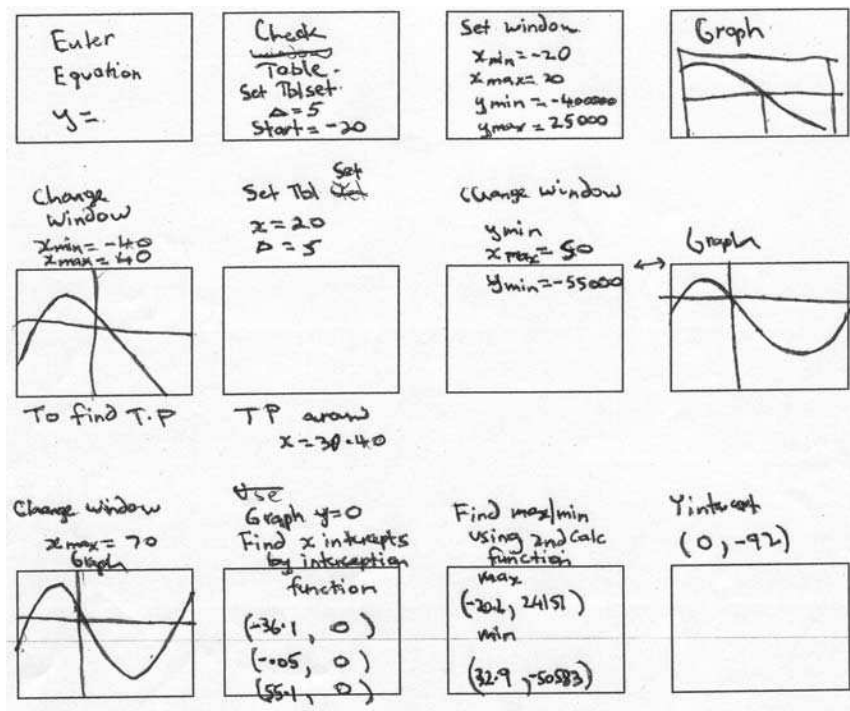


Figure 1. One teacher's solution to the task using the recording sheet.

Respondents were also asked to record the type and model of graphing calculator they usually used and to provide a self-assessment of their expertise in (a) using a graphing calculator and (b) teaching with a graphing calculator. Only 38 of the 42 teachers who handed in the record sheet responded to these questions. Texas Instruments (TI) calculators predominated with all but one teacher using a TI-83 or TI-83 Plus. The other respondent used a Casio CFX-9805GB Plus. These data are undoubtedly skewed by the collection being in Victoria where TI is the dominant supplier to the high school market.

For self assessment of their expertise with a graphing calculator using a 5 point Likert scale from 1 (*novice*) to 5 (*expert*), 63% of respondents rated themselves ≥ 4 and 89% rated themselves at least moderately skilled users (≥ 3). Only one participant used the novice category. Half the respondents rated themselves ≥ 4 in expertise in teaching with a graphing calculator, a figure rising to 87% rating themselves at least moderately skilled (≥ 3). A comparison of self-rating as user and teacher showed 76% of respondents gave the same rating for both categories. Seven respondents (18%) rated themselves higher as a user than as a teacher whereas two respondents (5%) rated themselves more highly as a teacher.

The Task

The task presented to the participants involved producing a complete graph for the cubic function $y = x^3 - 19x^2 - 1992x - 92$. Participants who did not have a graphing calculator were provided with a TI-83. Although from the setting participants may have inferred that a graphing calculator may help in their solution of the problem, they were asked to attempt the task using any methods they preferred.

The Analysis

The record sheet with its beginning response to the task was subjected to intensive response analysis. Commonalities in approach were noted particularly amongst (a) initial actions undertaken, (b) features of the graphing calculator used, (c) responses to particular views of the function, and (d) whether the teachers demonstrated explicit use of mathematical knowledge in conjunction with graphing calculator knowledge. Additional written comments by teachers regarding their own teaching approaches were also noted.

Initial View of the Function

As the result of initial actions is likely to have greatest effect on subsequent task progress, these actions were analysed in terms of the resulting view. Seven initial action categories and seven initial view categories as observed in the data will be described. There is not a one to one correspondence between the action and view categories. Some actions led to more than one possible view for the function and vice versa. Undertaking particular initial actions that led to each initial view were shown in previous work with students to have differing consequences for subsequent actions and ease of solution (Brown, 2003a).

The varied initial actions of respondents demonstrate the myriad approaches used by teachers in finding a global view of a given function in a graphing calculator environment. The initial overt action of respondents was classified into categories in ascending order of frequency of use as shown in brackets: editing the WINDOW settings (10); using Zoom Standard (on the TI this is $-10 < x < 10$, $-10 < y < 10$, and on the Casio CFX-9850GBPlus $-6.3 < x < 6.3$, $-3.1 < y < 3.1$) (9); exploring explicit mathematical information by hand, algebraically and/or graphically (6); direct selection of Zoom Fit (an automatic range scale feature for a given domain) (4); using TABLE (3); using GRAPH (uses previous WINDOW settings) (2); or editing WINDOW settings in conjunction with Zoom Fit (1). For a further seven respondents it could not be determined unequivocally which of Zoom Standard or GRAPH had been selected.

Initial windows seen by the users were classified into categories dependent on the visible aspects of the function or the position of which could easily be inferred. The categories were: a global view (GV) which showed all global aspects of the function

including all key features; an almost global view (AGV) where the majority of a global view was seen and the position of unseen features was immediately apparent; significant view (SV) where all or the majority of key features were unsighted, however, the function was clearly visible across the majority of the viewing domain; no apparent view (NAV) where no view of the function was obvious; apparent vertical lines (AVL); function coincident with y axis and no other part of the function visible (FCY); and unknown (neither window settings nor view recorded). Examples of the six known initial viewing window categories are shown in Figure 2. The number of respondents for each category were: GV $n = 7$, AGV $n = 1$, SV $n = 6$, NAV $n = 16$, AVL $n = 8$, FCY $n = 1$, and unknown $n = 2$. From a teaching perspective the first three categories should allow unproblematical progress towards a solution whilst the remainder are potentially problematic for students (Brown, 2003a). For this reason the following analysis begins with the initial view seen and considers actions that led to this view. The various paths to each of these categories will now be presented.

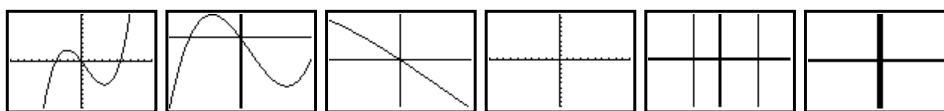


Figure 2. Examples of observed initial window view categories, GV, AGV, SV, NAV, AVL, and FCY.

Initial Window: A Global View. Of the seven respondents whose initial actions resulted in a global view, four adjusted the WINDOW settings whilst the others selected Zoom Fit. Within these two categories some respondents demonstrated use of additional mathematical knowledge to inform their selection. Of those teachers who adjusted the WINDOW settings, two determined information about the stationary points of the function prior to adjustment. One of these recorded, “thought process: cubic $\therefore$ 3 x ints [sic], 1 y int, 2 tp’s.” The derivative function of y_1 was then entered into y_2 and both functions graphed with WINDOW settings of $-200 \leq x \leq 200$, $-50000 \leq y \leq 30000$. All three respondents who initially selected Zoom Fit did so in conjunction with entering values for the viewing domain in the WINDOW settings. Two teachers, for example, used a graphing calculator program (Cubic) to find three solutions to the function and used these to inform selection of a viewing domain before accessing the autoscaling feature, Zoom Fit.

Initial Window: An Almost Global View. The one respondent who initially found an almost global view (AGV) followed a similar path to the last two teachers determining information about the x intercepts, and entering an informed choice for the viewing domain followed by selection of Zoom Fit. However, this teacher used the TABLE feature to identify two intercepts, between -30 and -40, and between -10 and 0, then indicated knowledge of the existence of a third “large & pos.” intercept before adjusting WINDOW settings and noting that they “should have looked further to the right.”

Initial Window: Significant View. Of the six respondents in this category four began with Zoom Fit. Two others made use of the TABLE feature prior to selecting their WINDOW settings. Two of the Zoom Fit users had the calculator set to the standard window. For two other teachers in the first group it can only be inferred that the viewing domain was set such that the x ordinates of the stationary points were excluded, thus giving a view similar to that seen for $-10 < x < 10$ although possibly over a different domain. One of the teachers in the second group used Table Setup and after observing the table values this teacher adjusted the WINDOW settings (see Figure 1). The remaining TABLE user

noted “use x between ± 10 , $y(-10) = 16928$, $y(10) = -20912$ and so adjusted the WINDOW settings. The resulting view drew the comment, “straight line! No TPs: probably 2.”

Initial Window: No Apparent View. The initial actions of 17 respondents resulted in no apparent view of the function being visible. All these respondents, except one, a Casio user with a different default window, saw their first view of the function with window values of $-10 < x < 10$ and $-10 < y < 10$. Nine respondents deliberately selected the standard window, whereas one simply pressed GRAPH but the previous window used had been the standard window. One teacher explained the rationale for this selection, “Graph, to see if I get anything on the screen. I don’t!” For the remaining seven teachers it is unclear which of GRAPH or Zoom Standard was selected, however, the resulting views were identical.

Initial Window: Apparent Vertical Lines. Eight respondents undertook initial actions that resulted in a function view apparently consisting of vertical lines. Seven of these adjusted the window settings, possibly informed by previous mathematical knowledge and the eighth selected GRAPH. All values selected were symmetrical about the axes ranging from $-40 < x < 40$ to $-100 < x < 100$ for the viewing domain and either $-100 < y < 100$ or $-1000 < y < 1000$ for the viewing range with the former being more frequently selected. The reaction of one of these teachers to the resulting view was “obvious that scale not appropriate.” One teacher attempted to use previous mathematical knowledge to inform the choice of window values, as they algebraically solved the derivative function for zero. Unfortunately the resulting choice of viewing range excluded any view of the stationary points. A second respondent also made explicit use of mathematical knowledge, recording the following comment next to their window values: “due to y intercept.”

Initial Window: Function Coincident with y axis. One respondent undertook initial actions that resulted in a view with the graph coincident with the y axis and no other portion visible. This teacher began by stating that the y intercept was -92 and proceeded to use a calculator program (Factor7) to find the factors but the program returned no factors. The teacher continued by “adjusting window settings in 1000s” to produce a view of the function coincident with the y axis (FCY).

Initial Window: Unable to be determined. Two teachers recorded none of the function views seen or window settings used. One provided an extensive list of possible actions to be undertaken; however, it is unclear if these were enacted. The remaining teacher suggested “trial values in window settings” and then “keep trialing until graph visible as cubic graph.”

Consequences of initial actions. As shown no one initial action can be guaranteed to provide a global view of the function. The application of mathematical knowledge also does not necessarily result in an initial global view. The use of the autoscaling feature appears to be significant in producing at least a significant view if not one that was an almost global or a global view of the function. In contrast, adjusting WINDOW settings may have resulted in a global view; however, for some respondents although this was an action undertaken in conjunction with mathematical knowledge, the results were a less than satisfactory view. Whilst teachers may seamlessly move from no view or an unusual view to a global view of the function, this is not necessarily the case for students (see Brown, 2003a) who have less mathematical knowledge and may have less graphing calculator knowledge than teachers.

Responses to Potentially Problematic Views

All seven respondents whose initial view was an Almost Global View or a Significant View were able to undertake subsequent actions resulting in a global view. The remaining

teachers' responses to problematic initial views are presented and considered with reference to the consequential view seen rather than the particular action undertaken. The potentially problematic responses of 68% of respondents whose initial view of the function was classified as No Apparent View, Apparent Vertical Lines, or Function Coincident with y axis are considered here.

Responses to No Apparent View. No actions undertaken in response to seeing no apparent view of the function immediately resulted in a Global View or Almost Global View. The actions undertaken by these sixteen respondents resulted in a Significant View ($n = 4$), No Apparent View ($n = 2$), Apparent Vertical Lines ($n = 6$), Function Coincident with y axis ($n = 1$), unknown ($n = 2$), and other view ($n = 2$) not described previously. Two examples of the last category are shown in Figure 3.



Figure 3. Views of the function classified as other.

Responses to NAV: Significant View. Four respondents undertook actions in response to No Apparent View that resulted in a Significant View. Three selected Zoom Fit. Of these three, one followed this immediately by zooming out near the origin, whereas two others adjusted the WINDOW settings. One teacher explicitly used mathematical knowledge to inform the choice of window values, “think about different scale. Try adjusting window, look at algebra”. They used the derivative to determine approximate turning points, “changed window” thus finding a global view. The response of the fourth teacher was to adjust the WINDOW settings informed by the TABLE. They noted “use table of values to help set window screen values”. This immediately resulted in an almost global view.

Responses to NAV: No Apparent View. Two respondents who subsequently observed No Apparent View of the function again adjusted the WINDOW settings. One appeared to apply knowledge of the y intercept as y minimum was adjusted to -100 . Mathematical knowledge was also used as the window domain was adjusted, first to $0 \leq x \leq 23$ (one of the factors for 92) then to $-23 \leq x \leq 0$. Both resulting views also showed no apparent view of the function. The WINDOW settings were adjusted to $0 \leq x \leq 100$ and $0 \leq y \leq -100$ (both $0 \leq y \leq 100$ or $-100 \leq y \leq 0$ show an apparent vertical line on the TI-83). This Casio user recorded a blank screen and noted, “I did not get any of the vertical lines you showed on the overhead. My screen was blank for these values. (However I have seen these on other occasions). My students’ response to this is often ‘My calculator is not working!’” The ROOT finder and TABLE were then used but it was noted that “I could have used TRACE to get this information.” Final window setting adjustment gave an almost global view.

Responses to NAV: Apparent Vertical Lines. Six respondents undertook actions resulting in the Apparent Vertical Line View. Five adjusted Window settings. The sixth selected Zoom Out and demonstrated a clear mental image of the features that the global view of the function needed to show. Adjusting window settings to $-100 \leq x \leq 100$ and $-100 \leq y \leq 100$ resulted in the vertical line effect. Re-editing these values to $-50 \leq x \leq 50$ and $-50000 \leq y \leq 25000$ gave an almost global view but subsequent use of Zoom Fit puzzled them, “but y int < 0 & pt infn. in 4th quad.” They produced a sketch with the y

intercept distinct from the origin and a point P labelled, noting “meant to be pt symmetry about P.” After two attempts at showing this point of symmetry, they finally sketched “to show other x int.” Another teacher’s comments provide some insight into his or her thinking—often a source of modelling. They began, “look at equation, what do I know, y intercept = -92, maybe I need a bigger window, -200, 200 on both x & y . I notice 2 ‘vertical lines’ $\therefore$ I need a larger ... scale to see turn pt. I then realised I have an intercept at -92 and can’t see it on my graph. I check my equation to see if it is typed correctly. It is. So I realise I need a larger vertical scale but smaller x so I check to see as $scl = 20$ what x dimensions I need.” WINDOW settings were adjusted. “Still I can’t see it well but I see the 3rd line knowing the basic shape [sketch of cubic with 2 stationary points shown] much much larger y scale.” Finally they comment “better but not enough” and then “hooray” as a global view resulted.

Responses to NAV: Other View. One teacher upon seeing no apparent view adjusted the WINDOW settings to $-1 \leq x \leq 1$ and $50 \leq y \leq 100$, resulting in the first of the views shown in Figure 3. Another teacher recorded the following, “Look at table. Note (0, -92) (as expected). Must be a soln b/w $x = -1$ and 0.” Adjustment of WINDOW settings to $-5 \leq x \leq 5$ and $-92 \leq y \leq 2000$ resulted in the second view shown in Figure 3.

Responses to Apparent Vertical Line View. Six teachers responded to the apparent vertical line view by adjusting WINDOW settings. Another two teachers selected Zoom Fit immediately finding a global or almost global view. The other six teachers had more diverse outcomes. Two adjusted the WINDOW settings, informed by knowledge of some key features and found an almost global view and global view of the function. The remaining four respondents had less successful outcomes and for three this resulted in a second apparent vertical line view. However, their response actions to this were not a repetition of previous action. One teacher, for example, recorded: “TRACE. Look at coord. To find TP Calc Zero, Table of Values. Window.” The result was an almost global view.

Responses to Function Coincident with y axis. On seeing a view of the function coincident with the y axis one teacher “used TRACE and looked at the coordinates to give a better idea of window settings.” Their subsequent adjustment to the WINDOW settings resulted in an almost global view of the function.

Conclusion: Some Implications for Teaching

A diverse range of approaches and actions were undertaken by the teachers. No one action, initial or otherwise, can be singled out as being superior, and therefore it is incumbent on teachers to, as Watson (2003) suggests, provide focused experiences for students to maximise their opportunities to develop expertise in such a range. Only then will students have sufficient tools, techniques, and flexibility in using these to overcome the diversity of dilemmas they could face and to limit possible views of the function observed. The approaches used in this study add to earlier findings of Brown (2003a, 2003b) that a combination of mathematical knowledge and graphing calculator knowledge facilitate a seamless solution for students and now teachers. An understanding of a range of graphing calculator features, flexibility in selection of these, and expertise in positioning any given view of the function portrayed in the graphing calculator window with one’s mental image of the function allow the user to successfully find a global view of the function. Use of the automatic scaling feature appears to support the finding of a global view and the production of a view that is considered appropriate by the users.

The view of Watson (2003) is that one responsibility of teachers is to increase learning opportunities by narrowing the range of what is possible to learn. She further suggests “the

activity of the learner is not necessarily the same as the teacher imagines setting” (p. 32). As the responses to the problem task discussed here demonstrate it may be difficult for the teacher to imagine the diversity of activity and consequential complexity of different learners to the same task. If the focus of learning is to successfully determine complete graphs of *difficult* functions in a graphing calculator environment both mathematical and graphing calculator knowledge are required, specifically (a) expertise and experience in positioning any given output of the graphing calculator with a mental image of a particular function and (b) knowledge of and experience in, using a range of graphing calculator features in conjunction with mathematical knowledge.

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