Coordinating Known and Unknown Quantities in a Multiplicative Context: Problem Conceptualization, Affordances and Constraints

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In line with continuing efforts to explain the demanding nature of multiplicative reasoning among middle-school students, this study explores the fine-grained knowledge elements that two pairs of 7th and 8th graders deployed in their attempt to coordinate the known and unknown quantities in the gear-wheel problem. Failure to conceptualize the multiplicative relation in reverse, mainly due to the numeric feature of the problem parameters and inherent inverse proportional relationship, led the students to use more primitive fallback strategies.

Multiplication and division situations have been analyzed from different orientations in mathematics education (Greer, 1992). For instance, Steffe (1994) analyzed children's multiplying schemes from the perspective of units-coordination. Vergnaud (1988) interpreted multiplication and division situations in terms of dimensional analysis. Thompson (1994) gives accounts of multiplicative reasoning in terms of quantities and quantitative relationships. Schwartz (1988) argues that multiplication and division are referent-transforming operations and distinguishes between extensive and intensive quantities. Nesher (1988) analyzed multiplicative relations in terms of the textual structure of problems. Fischbein et al. (1985) illustrate how implicitly-held models of multiplication and division lead to cognitive conflict with formal algorithmic structures. They assert that "people naturally tend to interpret facts and ideas in terms of structured models that are behaviorally and enactively meaningful" (p. 15). For example, the primitive model for multiplication is repeated addition.

In its basic form, a multiplication/division situation can be regarded from the perspective of a relation, mathematically equivalent to $a_1 \times a_2 = a_3$, where a_1, a_2, and a_3 are either integers, rational or real numbers. If, for instance, a_2 is unknown, one has to coordinate the known quantities a_1 and a_3 to determine a_2. Among others, the context and numeric features of a problem determines how such a multiplicative relation is established by the problem solver. In a previous study (Ramful & Olive, 2008), I analysed how two middle-school students articulate the multiplicative relation $w_1 d_1 = w_2 d_2$ in a balance beam context (where w_1, w_2, d_1 and d_2 represent the weights and distances on the two sides of the fulcrum). In the present study I analyse the multiplicative relation $n_s S = n_L L$ when two gears turn synchronously (where S, L, n_s, and n_L represent the number of teeth and number of turns of the two gear wheels). The main aim of this study was to understand how problem conceptualization enables or constrains students in reasoning multiplicatively. In that sense, I looked for those critical knowledge elements which opened solution paths or which hindered students from coordinating the quantities. I referred to the literature on multiplication/division, fraction, ratio, and proportion to identify those constraints that have been shown to affect students' ability to work with multiplicative structures. Vergnaud (1988) points out that:

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> the complexity of problems depends on the structure of the problem, on the context domain, on the numerical characteristics of the data, and on the presentation; but the meaning and the weight of these factors depend heavily on the cognitive level of the students. (p. 143)

Along the same lines, Kaput & West (1994) point out that there are four broad categories of task variables to take into account in problem solving situations: semantic structure, numerical structure, tools and representation, and the forms of the text. Fischbein et al. (1985) also list a similar set of factors: familiarity of context, quantities involved, size and type of numbers involved, relation between the situation referred to and the appropriate operation, rigidity effects associated with specific operations and intuitive intervening models. They found that the numerical features (e.g. decimals and size of operator and operand) of the data in a problem determined the choice of arithmetic operation as being either multiplication or division. Further, they argue that such intuitive models may often "slow down, divert or even block the solution process when contradictions emerge between the model and the solution algorithm" (p. 14). In addition, Harel, Post, & Behr (1988) showed that changing a divisor from an integer to a decimal may lead respondents to different interpretations of the same problem.

Theoretical Framework

To understand how the participants in this study conceptualized the problem situations, I used Vergnaud's (1988) idea of concepts-in-action and theorems-in-action. Vergnaud's theory provides a theoretical framework that permits the articulation between the mathematical problems to be solved, knowledge deployed, schemes, concepts and symbols involved in the solution procedure. Mathematical concepts exist in relation to each other and draw their meaning from a variety of situations. To analyze the complexity of the interrelatedness of concepts, Vergnaud (1988) introduced the theory of conceptual fields. He defines a conceptual field as "a set of situations, the mastering of which requires the mastery of several concepts of different natures" (p. 141). He considers the conceptual field of multiplicative structures as "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply and divide" (p. 141). This field can be regarded as consisting of a range of concepts and operations including multiplication, division, fraction, ratios, proportions which are referred as concepts-in-action. Apart from mathematical concepts, it also encompasses students' ideas (both competencies and misunderstanding), "procedures, problems, representations, objects, properties, and relationships that cannot be studied in isolation" (Lamon, 2007, p. 642). Vergnaud characterizes students' reasoning by symbolizing the different ways in which they articulate mathematical relations. He defines Theorems-in-action as the "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (1988, p. 144). Theorems-in-action are held to be true propositions and may even be flawed. They provide behavioral evidence of the mathematical relations that students may be using. The same mathematical relation may be interpreted through various theorems-in-action. Moreover, the term *affordance* is conventionally referred to the action potential (Bower, 2008) of an ICT-mediated environment. I use the term *affordance* to refer to a representation of a theorem-in-action that opens solution paths in the course of mathematical problem solving.

The Multiplicative Context: The Gear-Wheel Problem

I chose the gear-wheel problem (Lamon, 1999) as a multiplicative context to identify the concepts- and theorems-in-action that the four participants deployed as well as the conflicts that they encountered. Essentially, this problem consists of coordinating the number of teeth (S and L) and turns (n_s and n_L) as two gears with different number of teeth turn synchronously. This problem shares three features that make it suitable for studying multiplicative reasoning. Firstly, this multiplicative comparison situation is complex enough to probe students' thinking as it requires the coordination of 4 variables (S, L, n_s, n_L) which are in multiplicative relation, i.e., the four variables are related in terms of proportion and not additively. Secondly, it involves an inverse proportion relation between the number of teeth and the number of turns as can be inferred from $n_s S = n_L L$ or $n_s / n_L = L / S$ or $n_s : n_L = L : S$. Thirdly, it can be solved in a number of ways and offers possibilities to identify different theorems-in-action. In fact, the relation, $n_s S = n_L L$ may be interpreted in a myriad of ways (e.g., $n_s / n_L = L / S$, $n_s = (L \times n_L) / S$, $n_s = (L / S) \times n_L$ or in terms of a ratio or proportion), all of which are mathematically equivalent but not necessarily operationally from the perspective of mental processes. Further, the numeric feature of the problem parameters may allow other creative strategies, as the data in this study show.

Method

Data were collected from two pairs of students at grades 7 and 8 in an urban middle school. The grade 7 students (Aileen and Brian) were interviewed in May 2008 and January 2009 while it was only possible to interview the grade 8 students (Jeff and Eric) in May 2008. In some cases, observing the constraints that the students encountered, I interviewed them on the same task on two different days. In the initial phase of the clinical interview (Ginsburg, 1997) , I asked them to determine the number of turns that the small gear makes when the large gear makes only one turn and vice-versa (Table 1).

Table 1

Structure of Problems (S and L: number of teeth; n_s and n_L: number of turns)

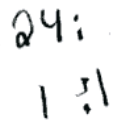
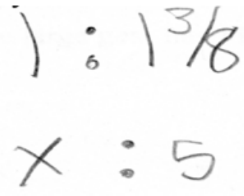
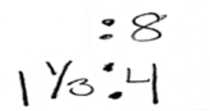
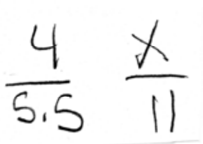
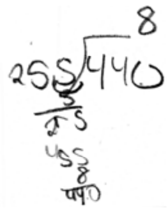
Problem	*Structure*	*Problem*	*Structure*
1	$S = 5,\ L = 10,\ n_s = 1,\ n_L = ?$	7	$S = 8,\ L = ?,\ n_s = 4,\ n_L = 1\frac{1}{3}$
	$S = 5,\ L = 10,\ n_s = ?,\ n_L = 1$	8	$S = 8,\ L = 11,\ n_s = 5,\ n_L = ?$
2	$S = 8,\ L = 11,\ n_s = 1,\ n_L = ?$	9	$S = 8,\ L = ?,\ n_s = 4,\ n_L = 1\frac{1}{3}$
	$S = 8,\ L = 11,\ n_s = ?,\ n_L = 1$	10	$S = ?,\ L = 11,\ n_s = 5\frac{1}{2},\ n_L = 4$
3	$S = 6,\ L = ?,\ n_s = 3,\ n_L = 1$	11	$S = ?,\ L = ?,\ n_s = 1,\ n_L = 2$
4	$S = ?,\ L = 24,\ n_s = 1\frac{1}{2},\ n_L = 1$	12	$S = ?,\ L = ?,\ n_s = 1\frac{1}{2},\ n_L = 1$
5	$S = ?,\ L = 36,\ n_s = 1\frac{1}{2},\ n_L = 1$	13	$S = ?,\ L = ?,\ n_s = 2\frac{3}{4},\ n_L = 1\frac{1}{4}$
6	$S = ?,\ L = 20,\ n_s = 1\frac{1}{4},\ n_L = 1$		

Gradually, the number of turns and teeth were increased and they were asked to coordinate the four parameters in the problem. I encouraged them to work in pairs and allowed them to solve the problem to the point where they were satisfied with their answer. I kept their written work as a trace of their thinking moment-by-moment. I used two video cameras so as to produce a restored view (Hall, 2000). In problems 3 to 10, three parameters were specified and the students had to determine the fourth unknown parameter by articulating the multiplicative relation. Problems 11 to 13 involved the determination of two unknowns, S and L, starting from two known quantities (n_s and n_L).

Results and Discussion

The four students differed in the ways in which they conceptualized and coordinated the number of teeth and number of turns in the gear-wheel problem. I characterize the way in which they conceptualized the problem in terms of the concepts- and theorems-in-action that they used consistently in Tables 2 and 3.

Table 2
Problem Conceptualisation by the 7th Graders

	Concept-in-action		Theorem-in-action
Aileen	She interpreted the number of teeth and number of turns in terms of two ratios (represented in the form of a proportion template). In each of the different situations she generated two ratios, one for the number of turns and one for the number of teeth.	*Problem 4*	
		Problem 8	 She subtracted $\frac{3}{8}$ from the 5 turns, compensating additively rather than multiplicatively
		Problem 9	
		Problem 10	 
Brian	He interpreted the problem as multiplication and division situation, and did not explicitly wrote a ratio to coordinate the quantities unlike his interviewer partner Aileen. He focused primarily on number of turns.	*Problem 4*	He interpreted $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$ and constructed one whole as $\frac{2}{3}$ of $1\frac{1}{2}$, i.e., $\frac{2}{3}$ of 24.
		Problem 8	He multiplied $1\frac{3}{8}\times 5$ which represents the number of turns that the small gear makes when the large gear makes 5 turns. He did not consider the inverse proportional relation.
		Problem 9	He multiplied 4×8 (i.e., 32) by $1\frac{1}{3}$ to obtain $42\frac{2}{3}$ operating $1\frac{1}{3}$ on the known quantity rather than the unknown quantity .

		Problem 10	He gave the following answer: "It will be 44 instead because this go by 4 turns. It started at 11, 11 times 4 is 44. So, if one has 44, this thing is going all around once. Then the other one have to have 5.5 times ... " and he did not proceed further.

Table 3

Problem Conceptualisation by the 8th Graders

Student	Concept-in-action		Theorem-in-action
Jeff	He interpreted most of the problem as a multiplication and division situation and focused on number of teeth to solve the problem.	*Problem 4*	$S = \frac{L \times n_L}{n_S}$
		Problem 8	$n_L = \frac{S \times n_S}{L}$
		Problem 9	He changed his consistent strategy from working with the number of teeth to the number of turns ($n_s = 4$ and $n_L = \frac{4}{3}$) due to the divisibility relationship between 4 and $\frac{4}{3}$.
		Problem 10	$S : \frac{L \times n_L}{n_S}$
Eric	He interpreted the problem as a multiplication and division situation and was occasionally constrained by problem context.	*Problem 4*	He gave the following justification: "Well, I did, I made 24, I made it into a fraction. I did 24 over 1 divided by 3 over 2 which is the same as 1.5. And I got 8 over 2 which is equal to 4" $\frac{3}{2} \div \frac{24}{1} = \frac{8}{2}$ 4
		Problem 8	In the first interview, his response suggests that he had no clear idea what he was doing. In the second interview, he multiplied $1\frac{3}{8}$ by 5 to obtain $\frac{55}{8}$ or $6\frac{7}{8}$ where he worked with the number of turns.
		Problem 9	He did not contribute much in the first interview. In second interview, the $\frac{1}{3}$ in $1\frac{1}{3}$ prompted Eric to divide 32 by 3.
		Problem 10	Following Jeff's lead, he attempted to divide 44 by 5.5.

While Jeff chose to work with the number of teeth, Aileen and Brian preferred to work with the number of turns and these called for different conceptualizations of the same situation. For instance, in problem 8 working with number of turns called for the theorem-in-action $(\frac{8}{11})\times 5$ while working with the number of teeth prompted Jeff to use the theorem-in-action $\frac{8\times 5}{11}$. Such differential conceptualizations generated different forms of conflict. Further, I captured the intuitive knowledge elements they deployed in their attempt to conceptualize the problems to open solution paths. For example, in problem 2, before the computation, Eric deduced that the small gear should make less than two turns: "Because I know the little gear had to at least make one turn but it wouldn't be like 2 complete turns." In problem 10, rather than dividing 44 by 5.5, Eric's strategy was to find *x* such that 5.5 multiplied by *x* gives 44. Instead of doing division involving decimals (5.5), he preferred to plug in numbers to find which number times 5.5 gives 44.

Constraint 1: Determination of the Unknown Quantity

The gear-wheel problem requires the coordination of known and unknown quantities in the multiplicative relation $n_s S = n_L L$. In some instances, the participants operated the given fraction on the known rather than the unknown. For example, in problem 9, Brian computed $1\frac{1}{3}$ of 32 rather than interpreting $1\frac{1}{3}$ of the unknown as being 32. It could also be observed that often the participants used the denominator of the given fractions as a pointer to determine the unknown quantity. For instance, in problem 4, Aileen used the fraction $\frac{1}{2}$ in $1\frac{1}{2}$ as a pointer to divide 24 by 2.

Constraint 2: The Inverse Proportional Relation between Number of Teeth and Turns

The conflict arising as a result of the inverse proportion between the number of turns and teeth could be observed at different instances of the interviews. Aileen and Brian (the seventh graders) and Eric (the eighth grader) encountered the same conflict in problem 6 ($11x = 8\times 5$). They all multiplied $1\frac{3}{8}$ by 5 instead of multiplying $\frac{8}{11}$ by 5. They did not take the inverse proportional relationship between number of teeth and turns into consideration at that instant This inverse proportional relation is analogous to measurement situations where the larger the size of the unit, the smaller the measure.

Constraint 3: Numeric Features of the Data

All four participants spontaneously solved problem 3, involving an integer ratio of turns (i.e., $n_s : n_L = 3:1$). However, Aileen, Brian and Eric were constrained to articulate the multiplicative relation in the other problems due to the non-integer ratio. The data show that students' capability to coordinate the known and unknown quantities in a multiplicative situation (mathematically equivalent to $a_1 \times a_2 = a_3$) is sensitive to the numeric feature of the data. When a_1 and a_3 are factors/multiples of the other, the students could readily find a_2, as could be observed in problem 3. However, when mixed numbers or improper fractions were involved the students experienced much difficulty. Consequently, estimation strategies were used to determine the unknown in the multiplicative relation. As highlighted in the review of the literature, Kaput & West (1994), Fischbein et al. (1985) and Vergnaud (1988) also reported such influence of numerical characteristics of problem parameters on students' ability to solve problems.

One may solve problem 8 using either number of turns in which case we use the ratio of turns that the small and large gear makes i.e., $1:\frac{8}{11}$ (involving the multiplicative comparison of fractional quantities) or in terms of ratio of teeth 8 : 11 (involving the multiplicative comparison of integer quantities). Jeff solved the problem using the number of teeth and as such worked with integer quantities. On the other hand, Aileen and Brian worked with the number of turns and encountered much difficulty in finding n_L. As is generally the case, problems involving fractional quantities are more demanding than those involving integer quantities. In addition, this study shows that even within fractional quantities the form of the number representation may influence problem solving behavior. The 'improper fraction' and 'mixed number' interpretation of rational numbers may offer different routes to solve the same problems. For instance, we observed that the same rational number $1\frac{1}{3}$ and $\frac{4}{3}$ cued different resources in problem 7, where the representation $\frac{4}{3}$ (interpreted as 4 thirds) allowed Jeff to observe the proportional relationship between 4 and $\frac{4}{3}$ to reverse his thought process.

In summary, this study shows that the four students solved the same problem using different strategies, deploying different resources and experienced different forms of conflict depending on the knowledge elements available to them at that point in time. The gear-wheel problem was conceptualized as a multiplicative comparison situation and as a ratio/proportion. However, they articulated these multiplicative relations with varying level of facility. The students were at different points in their development of multiplicative reasoning. On a continuum, I would consider Eric and Jeff at the two extremes, with Jeff at the upper end and Aileen and Brian in between. Neither Jeff nor Eric felt the necessity to use a proportion schema like Aileen. However, although she had the proportional schema, she was at times limited in using it. One of the fallback strategies that was observed is that students tend to make intuitive attempts, leaving behind mathematical principles in their attempt to solve problems. For example, in problem 8, to find *x* in her proportional schema $1:1\frac{3}{8}$ and $x:5$, Aileen subtracted $\frac{3}{8}$ from 5, thereby compensating additively rather than multiplicatively. An important observation from this study is that divisibility relationship among problem parameters is a critical factor that influences problem conceptualization.

In terms of instructional implications, this study suggests that middle-school students need to be given purposeful tasks to work with non-integer ratios and to work interchangeably between ratios and fractions. Similarly, such students need to be given opportunities to work with fractional units e.g., interpreting $1\frac{1}{2}$ as 3 units of $\frac{1}{2}$. More importantly, multiplicative comparison which lies at the basis of ratio and proportion situations, should be fostered especially in terms of fractional quantities.

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Strategies Used by Students to Compare Two Data Sets

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One of the common tasks of inferential statistics is to compare two data sets. Long before formal statistical procedures, however, students can be encouraged to make comparisons between data sets and therefore build up intuitive statistical reasoning. Such tasks also give meaning to the data collection students may do. This study describes the answers given by beginning university students to tasks involving comparing data sets in graphical form, originally designed for students between Grades 3 to 9. The results show that whereas all the students had successfully completed either pre-tertiary mathematics or a bridging mathematics course many had similar difficulties to students of a younger age. In particular, they did not use a measure of centre or proportional reasoning when appropriate.

One of the common tasks in inferential statistics is to compare two data sets. For example, is one group faster than the other group? Does the new drug work better? In the formal procedures of inferential statistics, questions similar to these are often answered by comparing the values of the arithmetic mean of each group while taking into account the value of the standard deviation of each group.

Using less formal means of making comparisons, however, students can compare two data sets by using a measure of centre such as the arithmetic mean or by using proportional reasoning. For students to use a measure of centre they need to know that this statistic is somehow representative of a group (Gal, Rothschild, & Wagner, 1990). Despite the wide spread use of the arithmetic mean (the *average*) in everyday applications, previous research has shown that students often only perceive the arithmetic mean as the learned algorithm. Because these students do not regard the arithmetic mean as a representative number they are generally unsuccessful in using it to make decisions about data (Mokros & Russell, 1995).

Gal, Rothschild, and Wagner (1989) investigated how primary students (Grades 3 and 6) compare two data sets. They found that most of the students in Grade 6 did not use the arithmetic mean in their solutions, even though they were familiar with its calculation. Many of the students used totals even when the data sets were not of equal size. They also found that many of the students in Grade 6 had difficulty in using proportional reasoning. In a later study Gal, Rothschild, and Wagner (1990) found that as students became older their understanding of the characteristics of the arithmetic mean improved but there was still a reluctance to use it as a tool to distinguish between two data sets. Whereas the formula for calculating the arithmetic mean was familiar to 2% of Grade 3, 61% of Grade 6, and 91% of Grade 9 students, the algorithm was applied by only 4%, 14% and 48% of the students respectively. They also did not generally use proportional reasoning or visual comparisons of the given graphical displays to reach their conclusions. Watson and Moritz (1998) also investigated students' thinking in comparing two data sets. In their study, 88 students from Grades 3 to 9 were given a series of tasks that required them to make comparisons between two data sets given in graphical form. Many of the students did not use the arithmetic mean in their conclusions, and those who did (10% of the Grade 6 students and 54% of the Grade 9 students) did not always do so successfully.

Another strategy in such tasks is to use proportional reasoning, which is valid when the groups are not of equal size. Proportional reasoning involves multiplicative reasoning instead of additive reasoning. For example, in answer to the question, "If green paint is