Problem Categorisation in Ratio – A Closer Look

Norhuda Musa
Tampines Primary School
<norhuda_musa@moe.edu.sg>

John Malone
Curtin University
<j.malone@curtin.edu.au>

This paper reports further findings from a study on categorisation of typical problems in the learning of ratio originally presented at the 2011 conference. The paper focuses specifically on the seven categories identified; examines strategies used by two classes of Primary 6 students to solve problems, and presents data in the form of student journal writing; interviews; voice recordings and a pupil perception survey. Pre- and post-tests were administered. Outcomes reveal the positive effects of learning ratio through categorisation.

This on-going study was first described at the 2011 *Mathematics Education Research Group of Australasia* (MERGA) conference. That paper examined the potential of categorisation of ratio word problems in order to enhance students' understanding of the mathematics of ratio (Musa & Malone, 2011). Principles such as *Cognitively Guided Instruction* (CGI) and *Case Base Reasoning* (CBR) were tied together to bring about optimal learning. CBR relies heavily on prior knowledge and appears to work well with CGI problem categorisation. In this process, problems are placed in various categories based on the distinctive feature each structure offers. Each category influences the strategies that pupils use to solve problems; hence, these categories are labelled not only based on their structure of questions but also on the concepts used to solve the problems. This current paper examines the seven categories in greater depth and describes the outcomes of the study to date.

Literature Review

Model Method, an innovation in the teaching and learning of mathematics in Singapore, was first introduced at the primary four level in 1983 to address the issue of pupils having great difficulty with word problems (Kho, Yeo & Lim, 2009), and it has become one of the key features of the country's mathematics curriculum. The theoretical basis for the model method comes from the second form of representation of Jerome Bruner's theory (1961), namely the Iconic mode. This mode is pictorial in nature and is in line with the Concrete-Pictorial-Abstract instructional approach. The model method involves drawing rectangular bars drawn in proportion, with known and unknown information or "generators" (Ng, 2004) fitted in. Different types of dotted lines are used to facilitate changes. Ng (2004) defines two stages to learning model construction: developing part-whole concept at primary one and two levels, and then moving on to constructing models based on proportional reasoning at primary three level. In both cases, identifying the generator, abstracting and articulating the technical relationships between parts and the generator are crucial to problem solving via the model method. Yeap (2010) attributed success among average pupils in TIMSS and Grade 6 national test to the use of the model method, which has "no doubt been helping to make challenging mathematics accessible to average students" (p. 89).

In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia). Singapore: MERGA.

This contention is certainly true – to a certain extent. This is because when problems become more complex, construction of models becomes a real challenge for the low and middle-scorer students. Pupils face difficulty in the basic application of model method – deciding on the correct type of models – in particular those involving part-whole, comparative, before and after and working backwards procedures. Furthermore, when not drawn in proportion, the model method fails to reveal the relationships between parts. Goh (2009) reported that despite its effectiveness in solving many challenging arithmetic word problems and its emphasis in classroom teaching and learning, many pupils still experience varying degree of difficulty in using the model method. The difficulties upper primary pupils faced are more apparent as increasing number of complex multiple step arithmetic word problems are attempted, which often require more complex model drawing.

Lo, Watanabe and Cai (2004) stated that Asian textbooks clearly distinguish ratio as a multiplicative relationship between two quantities, complete with examples and exercises to enhance the learning of ratio. However, in Singapore, textbooks do not challenge pupils with problem variety (Ho, Teong & Hedberg, 2004). Hence, relying on textbooks alone is insufficient as a "culture of challenge has developed in Singapore mathematics classroom" (Yeap, 2008, p.1). With a significant proportion of Primary School Leaving Examination (PSLE) items being challenging, it is understood that every pupil is expected to do challenging problems. Prime Minister Goh, in 2004, made the call for teachers to teach less and use the available time to "excite pupils in the learning process" and allow them to learn more. As a result, alternative strategies are encouraged for pupils who are not performing well. Categorisation is a new alternative that offers conceptual understanding of the fundamentals of ratio mathematics.

Middleton and Van den Heuvel-Panhuizen (1995) noted that many pupils have a "high degree of proficiency at arithmetic computation but sometimes with little conceptual understanding". Yeap (2008, p. 7) concurs with this, stating that the "ability to select the correct operations and perform computations do not always lead to a correct solution". Categorisation will address the issue of little conceptual understanding as pupils will be dealing with concepts before deciding on the categories. Interestingly, Chick (2010, p. 2) stated that in studies conducted by Mitchelmore, White and McMaster (2007) and Steinthorsdottir and Sriraman (2009) the researchers showed how students' knowledge can develop with appropriate CGI and identified levels of understanding, including one associated with the difficulties of dealing with non-integer multipliers. It is anticipated that with categorisation, the issues raised can be addressed so that more meaningful learning of ratio can occur.

Method

This study is being carried out in two classes, each of thirty-two Primary 6 pupils in a primary school in Singapore. They form a non-random purposive sampling group. Pupils in the study experience no problem with multiplying and finding equivalent ratios: They were first exposed to ratio in Primary 5; they possess basic knowledge of procedures and are expected to use considerable reasoning in solving problems. It is through this reasoning that pupils are expected to discover the concepts tied to the categories.

Two teachers also participated in this mixed methods approach where quantitative and qualitative data were simultaneously collected and merged. Creswell (2008) believes that the strengths of one data form offset the weaknesses of the other form. A pre-test consisting of seven questions, each from the seven categories, was administered prior to the commencement of the study and data in the form of MP3 recording, interviews (between teacher and pupils) and journal writing have been carried out at regular intervals throughout the study's duration.

The teachers led the pupils into the categories using the CGI and CBR frameworks and facilitated pupils' learning by scaffolding the learning of the concepts in each category. At the end of the intervention of the seven categories, pupils' are expected to be able to identify the categories on their own. Details on the seven categories follow:

Category 1: Values Assigned

Example: Sam poured lemon juice for her three friends in the ratio 3 : 4 : 8. The smallest share was 250 ml less than the biggest share. How many millilitres of lemon juice did she pour altogether?

Concept: Ratio parts are assigned to values.

Category 2:One quantity remains constant

Example :Three classes were asked to make cards. Class A made $\frac{4}{5}$ of the number of cards made by Class B. Class C made twice the number of cards made by Class A. Class C made 72 cards more than Class B. How many cards did the 3 classes make altogether?

Concept: One quantity in the ratio remains constant. In this category, three values are compared; A to B and C to A. Pupils are guided to "see" that the value of A should be the same when compared to different quantities like B and C. The idea of reduced ratio is discussed here, as the difference of A is 4 units in the first ratio and 1 unit in the second ratio, comes about because of reduced ratios. 'Generalising' (Ng, 2004, pg 43) will take place at the end of the lesson to ensure pupils are able to identify the pattern (concept) that is consistent of the idea.

Category 3:Constant Difference

Example: A baker makes a total of 120 pies and muffins in the ratio 3 : 5. After he sold an equal number of pies and muffins, the number of pies and muffins left was in the ratio 3 : 8. How many pies and muffins did he sell altogether?

Concept: Difference between parts must remain constant when equal amounts are removed, added.

As pupils become aware that one quantity remained constant in category 2, they are guided through the CBR framework where the old case (category 2) is used to solve the new case (category 3). The number of pies and muffins are changing, but the difference between them remains constant (see Figure 1). The teacher uses Socratic prompts in order to probe thinking at a deep level to bring about that idea. Examples using 'specific numerical ratio' (Chick, 2010) will be added in to promote understanding.

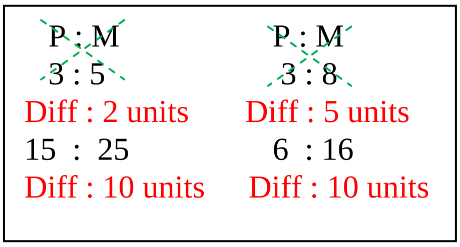


Figure 1.

Following the introduction of Categories 1,2 and 3, pupils were familiar with the CBR approach requiring them to solve particular question types using prior knowledge.

<u>Category 4</u>: Total remains constant

Example: Three boys, Aaron, Ben and Charlie shared the cost of a birthday present for their father. The ratio of Aaron's share to the total of Ben's and Charlie's share was 1:3. The ratio of Ben's share to the total of Aaron's and Charlie's shares was 1 : 5. Charlie paid $50 more than Ben. Find the total cost of the present.

<u>Concept</u>: Total, when compared to different quantities in a question, remain constant. Here, if the two ratios are compared, pupils will see that sum of each ratio is actually the cost of the present. It is important to note that 1 unit in the first ratio is not equivalent to 1 unit in the second ratio. Pupils will have to find a way to make them equal (see Figure 2). Pupils will then use the parts to solve the problem.

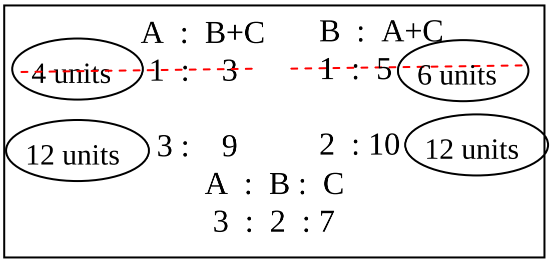


Figure 2.

<u>Category 5</u>: Fractional parts of ratio

Example: Danny, Kendrick and Teddy shared some sweets in the ratio 3 : 6 : 4. Danny kept $\frac{1}{5}$ of his sweets and gave the rest of his sweets to Kendrick and Teddy in the ratio 1 : 3. As a result, Kendrick had 700 sweets more than Teddy. How many sweets did Danny keep for himself?

<u>Concept</u>: Common multiples are used to avoid working with fractions.
Pupils will have to take one fifth of 3 units out, which will leave them with a mixed fraction. To avoid this, the idea of common multiples is used. So instead of working with 3 : 6 : 4, pupils will obtain the equivalent ratio 15 : 30 : 20. This way, pupils are able to take one fifth of 15 units, leaving them to work with integers.

<u>Category 6</u>: Ratio of changing quantities

Example: The ratio of the number of Daniel's pens to the number of Eunice's pens was 1 : 5. Then Daniel bought 12 more pens and Eunice bought 17 more pens. The ratio of the number of Daniel's pens to the number of Eunice's pens became 1 : 4. How many pens did Daniel have first?

<u>Concept</u>:
Use of "set"and 'replacement'approaches as all quantities are changing.This category involves theuse of algebra. Since pupils have not learnt algebra, the heuristic called 'Sets' is used instead (see Figure 3). Pupils do not realise that by using the 'Set' approach, they are actually solving simultaneous equations.

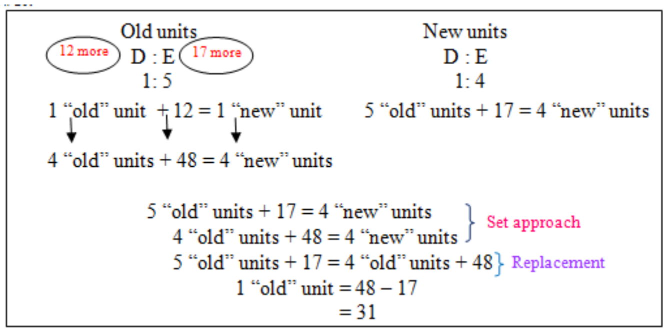


Figure 3.

Category 7: Ratio of a ratio

Example: At a game stall, every child needed 4 tokens to exchange for a prize, while an adult needed 5 tokens. Given that $\frac{2}{3}$ of the people who exchanged their tokens for prizes were children and total of 1092 tokens were collected by the game stall, how many tokens were collected from the adults?

Concept: The "hidden" ratio is multiplied to the "given" ratio. In this category, there are two ratios; one with definite value, "hidden ratio" (number of tokens each person gets) and the other with indefinite value, "given ratio" (the number of people) (Figure 4). The quantities in the two ratios are multiplied together. The pre test was administered again as a post test at the conclusion of the intervention.

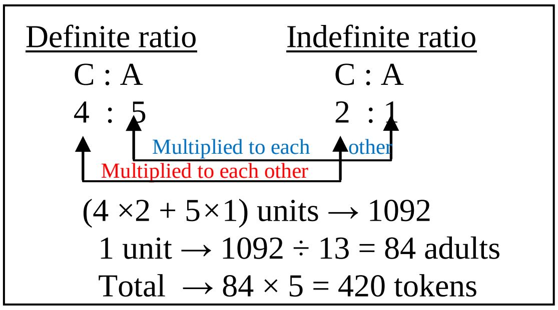


Figure 4.

Findings

Changes in Student Performance

Category	Pre Test (64 pupils)		Post Test (64 pupils)	
Category 1	64	100%	64	100%
Category 2	59	92.1%	64	100%
Category 3	29	45.3%	64	100%
Category 4	24	37.5%	58	90.6%
Category 5	18	28.1 %	60	93.7%
Category 6	15	23.4%	42	65.6%
Category 7	23	35.9%	52	81.2%

Changes in Student Thinking

Interviews: The responses of five of the pupils interviewed are reported here. All mentioned that they had to rely on their own knowledge because their teacher "did very little explaining". This indirectly "forced" them to read the question for understanding. At times when the problem became too difficult, they admitted that understanding was a problem. Since they were in the culture of having to read the question repeatedly, comprehending the problem became less difficult. Although their solutions were incomplete, they were at least able to transform some of the clues in the problem into simple, correct mathematical forms. Based on their work, it was noticeable that they no longer wrote "incorrect direct translations into number sentence based on random numerical cues (Goh, 2009). This appears to be an excellent way to consolidate the first stage of problem solving: reading and understanding.

Four pupils agreed that there was a positive change in thinking. They found themselves engaged in thinking unlike before where they had little idea of where to begin. They said that categorisation has given them a form of "picture" to help them 'kick start' their thinking, although one pupil felt that categorisation did not benefit her. She found it time consuming to identify categories. She preferred to "just see the question and do it" because that had worked well for her.

When asked if it helped that a strategy was attached to a category, three of them agreed that was the case. Although the school had been exposing them to a wide variety of heuristics, they were not able to choose the appropriate one to use. After the categorisation intervention, they found themselves "more aware" of how each strategy was to be used. One of them found it useful only for the last two categories as he "had no idea of what was required" of the question. The other pupil felt that it was unnecessary as she felt that she knew exactly what each problem entailed.

Voice recording through MP3 player:

Transcripts of pupils' thought processes during pre- and post-tests were compared. It was found that there was a noticeable change in student thinking in the post test; many pupils were able to reason logically and correctly. Hence they were able to categorise questions correctly. Following the introduction of Categories 1, 2 and 3 the thought processes revealed that all students in the two classes (64/64) managed to reason correctly, placing problems in the correct categories and obtaining correct solutions.

Later, 90.6% of the pupils (58/64) managed to identify the category for the given problem in Category 4. Three of the other six pupils identified the category incorrectly, while the rest made some calculation mistakes. In Category 5, pupils' reasoning was very good, with almost everyone doing well in this category. In the MP3 recording every pupil mentioned the use of 'lowest common multiple'.

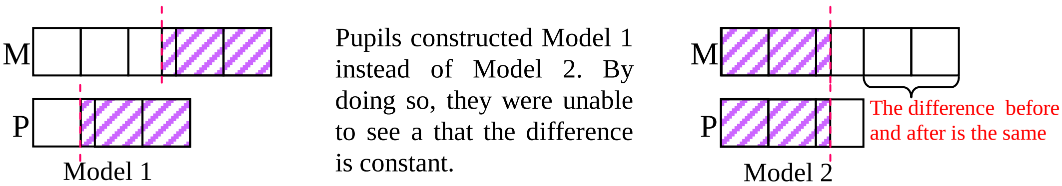
For Category 6, 96.8% (62/64) of the pupils were able to identify the correct category for category 6 problem and 65.6% (42/64) managed to obtain the correct solution. One particular reason for this was because the category involves creating "sets" or equations. Those who managed to get this completely correct used basic algebra to solve this. The rest did it by algebra too, but had difficulties halfway through the working. When MP3 recordings of 12 pupils who failed to get this question correct were played, it was discovered that all of them could identify the correct category and the concept, but were confused when it came to the technical part of algebra – they could not manage when they transposed to the other side of the equation; they were unable to "undo" the operations

(Mason, 1988: Driscoll, 1999 in Ng, 2004). They worked through the equation using their understanding of equivalence and constructed their knowledge based on intuitive problem solving strategies.

In Category 7, members of one class were able to identify this category as the only one with two ratios that did not refer to the same quantity. The other class reasoned it differently; pupils said that of the two ratios, one of them had a “definite value” and the other an “indefinite value”, and that 1 unit should be multiplied to quantities found in the indefinite ratio. In the MP3 recording of the pre-test, 22 pupils from the first class which mentioned the existence of two ratios referring to different quantities could not find the other ratio C : A = 4 : 5. It did not cross their minds that the tokens each received could be written as a ratio. Because of this, they were not able to obtain a solution. Interestingly, 96.8% (31/32) of pupils from the second class were quick to identify the “definite” ratio and managed to pull through the solution quickly and correctly. Different reasons came about from the two classes as the teachers were instructed to let the pupils discover them on their own. More scaffolding was done to the first class to help address the gap between students’ initial ability to understand concepts and the new concepts they had to learn in Category 7.

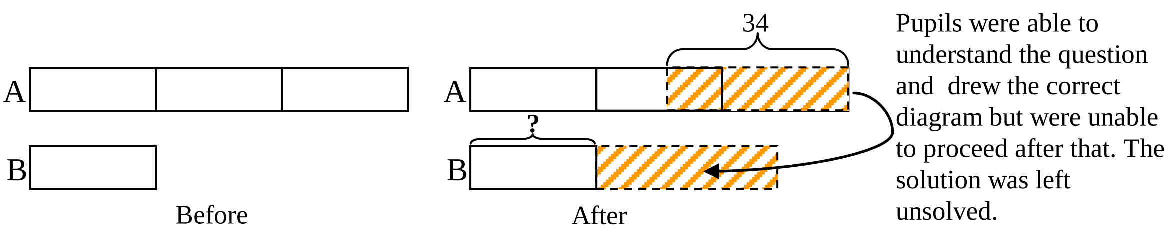
Strategies Used

In Category 3, pupils who solved using the model method experienced difficulty with the construction of the model. Most left their model drawing as shown in Model 1. They were unable to generate a solution as the method failed to reveal relationships between parts.



These pupils understood the question but they were not able to transform the information in the text into mathematical forms (Newman, 1983). Another mistake commonly made in model drawing is found in Category 4, where a change in situation is involved. Those who did the model method were stuck at the planning stage and were unable to construct the model. They left this question with an incomplete model drawing. An example from a similar problem found in their daily work in class is as follows:

Ratio of Ally’s marbles to Bill’s marbles is 3:1. Ally gave 34 marbles to Bill and found that she had $\frac{4}{7}$ as many marbles as Bill. How many marbles did Bill have at first?



Those who did by categorisation did well for this question (Figure 5).They were able to identify the correct concept which led them to a solution. This approach may seem very abstract, but it *does* help those who have difficulties in constructing model drawings involving change. During one of the classroom sessions, pupils were given this question which did not belong to any specific category. It was given to them to see how they respond to it:

> In classes 6A and 6B, the total number of girls was twice the total number of boys. The ratio of boys to girls in 6A was3 : 4 and the ratio of boys to girls in class 6B was 1 : 6. If there were 8 more girls in class 6A than 6B, find the number of pupils in class 6A.

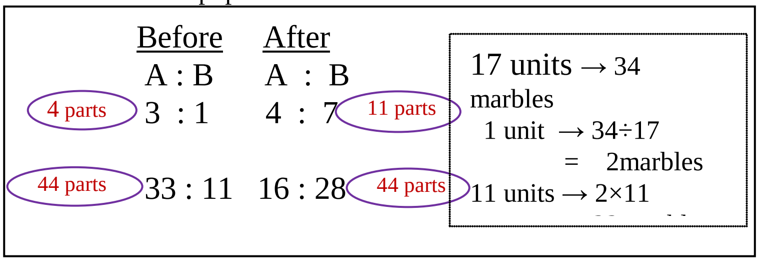


Figure 5.

It was interesting to see the pupils trying to fit this into the seven categories. A group of pupils decided that "This cannot be categorised. We need a new category". So based on their solution, this particular group worked out the solution and named this category "Conditional Ratio" because according to them, they solved it using the conditions that were given in the question. They adopted the "listing" heuristic. Fig 6 shows their working.

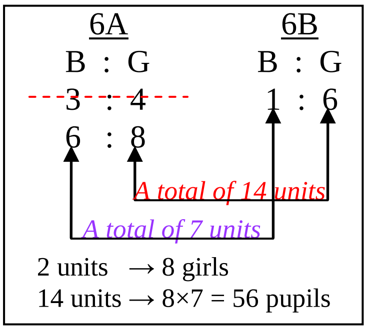


Figure 6.

Journal Writing

In journal writing, pupils were asked to explain briefly to their parents what categorisation was all about. Two pupils took an interesting approach (shown in Figures 7a and 7b). They started a question and varied the categories simply by changing numbers and context. This was impressive and encouraging as it showed that not only had categorisation impacted on these two, but it had also empowered them to create questions based on the categories.

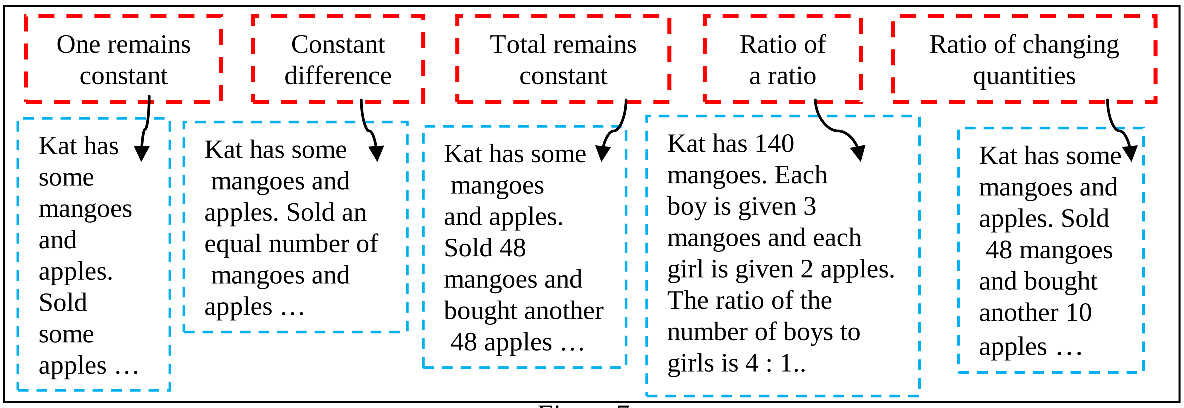


Figure 7a.

When 20 red pens were replaced by 20 blue pens ……. Constant total
When 20 red pens were replaced by 10 blue pens ……. Ratio of changing quantities
When 20 red pens and 20 blue pens were removed ……. Constant difference
When 20 red pens were removed ……. One remains constant

Figure 7b.

Conclusion

Categorisation was formally introduced last year at the Primary Six level. The teachers and pupils responded positively to categorisation. More importantly, categorisation appears to enable pupils to reason logically; analyse mathematical situations and construct logical argument. In order to do this, pupils must first read the problem and understand it. Only then can they devise a plan to solve the problem. Categorisation shows promise for assisting students towards a better understanding of learning the mathematics of ratio.

References

Chick, H. L.(2010). Aspects of teachers' knowledge for helping students learn about ratio. In L. Sparrow, B. Kissane & C. Hurst (Eds.) *Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia*. (pp. 131-139). Fremantle : MERGA.

Creswell, J. W. (2008). *Educational research: Planning, conducting and evaluating quantitative and qualitative research* (3rd edn.). Pearson. Merill Prentice Hall.

Goh, S. P.(2009). Primary five pupils difficulties in using the model method for solving complex relational word problems. *Unpublished Master of Education thesis* submitted to Nanyang Technological University, Singapore.

Ho, K. F., Teong, S. K., & Hedberg, J. G. (2004). *Creating problem solving repertoires*. Centre for Research in Pedagogy and Practice, Nanyang Technological University, Singapore.

Kho, T. H., Yeo, S.M. & Lim, J. (2009). *The Singapore Model Method for Learning Mathematics*. Curriculum Planning and Development Division. Ministry of Education, Singapore : EPB Pan Pacific.

Lo, J. J., Watanabe, T. & Cai, J (2004). The development of ratio concepts : An Asian Perspective. *Mathematics Teaching in Middle School*, 9(7), 362 – 367.

Middleton, J. A.& van den Huevel – Panhuizen, M. (1995). *The Ratio Table. Mathematics Teaching in the Middle School*. I (4), 282 – 288.

Musa, N. & Malone, J. A. (2011). The use of problem categorisation in the learning of ratio. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.) *Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia and the Australian Association of Mathematics Teachers*. (pp.565-574). Adelaide: AAMT and MERGA.

Newman, A. (1983). *The Newman Language of Mathematics Kit – Strategies for Diagnosis and Remediation*. Sydney, Australia: Harcourt Brace Jovanovich Group.

Ng, S.F. (2004). Developing Algebraic Thinking in Early Grades : Case Study of the Singapore Primary Mathematics Curriculum. *The Mathematics Educator*. *8* (1), 39 – 59.

Yeap, B. H. (2008). *Challenging Mathematics in Primary School National Examination in Singapore.* www.amt.canberra.edu/icmis16psinyeap.pdf

Yeap, B. H. (2010). *Bar Modeling. A problem solving tool. From research to practice. An effective Singapore mathematics strategy*. Marshall Cavendish. Singapore