Developing Pedagogical Strategies to Promote Structural Thinking in Early Mathematics

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The Pattern and Structure Mathematical Awareness Program (PASMAP) is an early mathematics program designed to promote structural thinking. PASMAP pedagogy removes the structure commonly provided for students in order to challenge them to construct their own, focusing student attention on spatial and numerical patterns and leading them to formulate their own generalisations. In this paper, we give some examples of Kindergarten PASMAP tasks that develop and link the themes of grid patterns, number patterns, multiplication and base ten numeration. Work samples drawn from a recent evaluation study are used to illustrate the range of students' structural development.

In our recent MERGA reports, we described a large-scale evaluation study of the effectiveness of the Pattern and Structure Mathematical Awareness Program (PASMAP) on Kindergarten students' mathematical development (Mulligan, English, Mitchelmore, Welsby, & Crevensten, 2011a, 2011b). This intervention program provided explicit and connected structured tasks over three school terms found to enhance the structural development of patterning and unitising, spatial structuring, base ten and multiplicative reasoning, and mathematical generalisations. This paper illustrates how an emphasis on pattern and structure shaped pedagogical strategies aimed at promoting simple generalisations.

Mason, Stephens and Watson (2009) take *mathematical structure* to mean the identification of general properties that are instantiated in particular situations as relationships between elements or subsets of elements of a set. Their view of structural thinking focuses on an important distinction between simply recognising elements or properties of a relationship, and having a deeper awareness of how those properties are used, explicated and connected. Some pedagogical strategies for promoting such awareness are described in Mason, Drury, and Bills (2007).

Mulligan and Mitchelmore (2009) examine the bases of structural development in early mathematics learning, describing the salient underlying features of pattern and structure common to a range of mathematical concepts. We take *pattern* to mean any predictable regularity in our environment, and *structure* to mean the way the pattern is organised. Most patterns involve number or space, and spatial structuring is critical to establishing relationships between patterns and structures. To quote Warren (2005, p. 305), "Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning".

Much recent research has focussed on children's early learning about mathematical structure. The importance of patterning skills, analogical reasoning and the development of structural thinking has been confirmed in several studies (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; English, 2004; Papic, Mulligan, & Mitchelmore, 2011). For example, the Dutch "Curious Minds" project highlights patterning and spatial skills moving beyond early numeracy (van Nes & de Lange, 2007). There is also increasing evidence that early algebraic thinking develops from the ability to see and represent patterns and relationships such as equivalence and functional thinking in early childhood (Warren & Cooper, 2008).

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A further line of research has shown that data modelling, a developmental process that begins with young children's inquiries and investigations of meaningful phenomena (Lehrer & Schauble, 2005) also requires children to seek structure and recognise patterns. Preliminary findings of a longitudinal study of data modelling in Grade 1 (English, 2010) indicate that children as young as six years old can successfully collect, represent, interpret, communicate, and argue about the structure of data provided they address familiar themes.

Over the past 10 years, we have accumulated evidence (Mulligan, 2010) that children's responses to a wide variety of mathematical tasks can be reliably classified into the following five levels of structural development:

1. *Prestructural*. Children pick on particular features that appeal to them but are often irrelevant to the underlying mathematical concept.
2. *Emergent*. Children recognise some relevant features, but are unable to organise them appropriately.
3. *Partial structural*. Children recognise most relevant features of the structure, but their representations are inaccurate or incomplete.
4. *Structural*. Children correctly represent the given structure.
5. *Advanced*. Children recognise the generality of the structure.

Our studies led us to conjecture that initial recognition of similarities and differences in mathematical representations plays a critical role in the development of pattern and structure, abstraction and generalisation. The development of multiplicative concepts (including understanding the base ten system, grouping and partitioning) is integral to building structural relationships in early mathematics. Spatial structuring is necessary in visualising and organising these structures (Battista, 1999).

Moreover, we have shown that children tend to respond at the same level on these various tasks (Mulligan & Mitchelmore, 2009). We have taken this finding to be evidence for a general characteristic that we have called Awareness of Mathematical Pattern and Structure (AMPS). We believe that AMPS has two components: not only an understanding of common mathematical structures but also a tendency to look for patterns in new situations. Crucially, children with a high level of AMPS tend to do well at mathematics, while those with a low level of AMPS tend to struggle.

The Pattern and Structure Mathematical Awareness Program (PASMAP)

PASMAP was designed on the assumption that AMPS was not an innate unalterable trait but a characteristic that could be developed through appropriate instruction. Several studies (summarised in Mulligan, 2010) have confirmed this hypothesis from as early as preschool (Papic, Mulligan, & Mitchelmore, 2011).

The program focuses on fundamental processes such as rhythmic and perceptual counting, simple and complex repetitions, growing patterns and functions, unitising and partitioning, grids and arrays, symmetry and transformations, congruence and similarity, and data modelling. For further details, see Mulligan, Mitchelmore, English, and Robertson (2010).

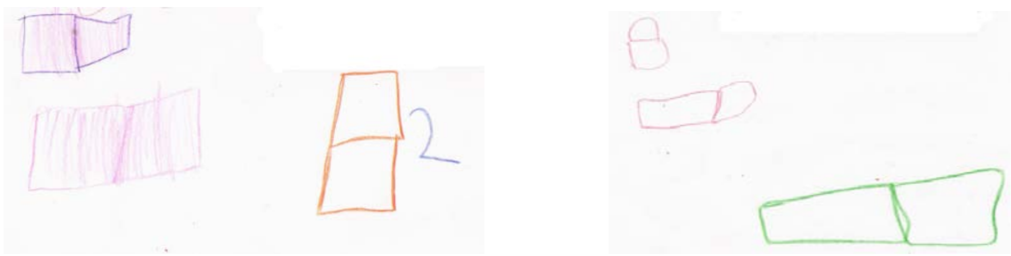
In PASMAP, children are encouraged to seek out and represent pattern and structure across different concepts and to transfer this awareness to other concepts. In other words, the aim is to promote generalisation in early mathematical thinking. This aim is achieved through *pattern-eliciting tasks* that require students to copy or reproduce a model or other representation. In the PASMAP pedagogy, the teacher uses probing questions to highlight important features of their drawings, to compare them with the model or with other children's drawings, and to focus their attention on similarities and differences in crucial aspects of spatial and numerical structure. Tasks are modified and repeated regularly, reinforcing and extending generalisations and providing links to prior learning. Further details of this pedagogy are to be found in Mulligan (2011).

Some Examples of PASMAP Pattern-Eliciting Tasks

To illustrate PASMAP pedagogy, we have chosen examples to illustrate how awareness of the structure of the rectangular grid is developed in Kindergarten. Some student work samples have been drawn from the evaluation study referred to earlier (Mulligan et al., 2011a, 2011b). In this study, eight classes from four schools in Brisbane and Sydney were taught using PASMAP for their entire Kindergarten year and a further eight classes in the same schools acted as a comparison group. In the PASMAP classes, five high-ability and five low-ability students, as measured by *I Can Do Maths* test (Doig & de Lemos, 2000), were chosen for closer study: students were videotaped and their work samples were collected. We will show some work samples from Heela, a high-ability student, and Lateh, a low-ability student. Both names are synonyms.

Theme 1: Spatial Grid Structure

After a sequence of tasks focused on simple repetition and spatial patterns (Papic et al., 2011), designed to ease children into the program, there is a focus on constructing and analysing simple grids. In the first of these, children are shown a 2 × 1 grid for a few seconds and then asked to draw it. The teacher then gives them a 2 × 1 grid and two matching squares and asks how many squares are needed to cover the grid. Different strategies for placing the squares are discussed, and students are also asked to fold the grid to explore the structure. The teacher then asks, “What’s the same?” and “What’s the different?” and students encounter ideas such as counting, shape, sides and vertices, rotation (turning), congruence (same size and shape), and fractions (half). The grid and squares are then removed and children draw the grid from memory in both horizontal and vertical orientations. After sharing and discussing their drawings, the class summarises what they have learnt and looks for links to their earlier tasks (e.g., in the towers they had made from unifix cubes). This may seem a very elementary task, but it is fundamental and many students found it quite challenging (see Figure 1).



Heela   Lateh

Figure 1. Two contrasting representations of a 2 × 1 grid.

The next lesson moves on to 2 × 2 grids (called “windows”), following a similar procedure. Previous ideas are reviewed and extended, and further ideas of rows and columns, clockwise and anticlockwise, vertical and horizontal, diagonals, and even quarters are encountered. The difference between the high- and low-ability children already becomes apparent, and indicates to the teacher how perceptive some children are in terms of recognising structural features while others pay little or no attention to mathematical features. Figure 1 shows two such contrasting drawings. Heela has already recognised that she does not need to draw separate squares, whereas Lateh is still struggling to draw congruent squares in the standard orientation.

Heela Lateh

Figure 2. Two contrasting representations of a 2 × 2 grid.

In subsequent lessons, the task is extended to larger rectangles. By repeatedly looking at what is the same and what is different between a given grid and their drawings, and by seeking generalisations from their observations, children gradually learn that a grid can be drawn using equally spaced, perpendicular lines. Additional tasks include making a sequence of squares of increasing size from 60 square tiles. Each activity reinforces the basic generalisation that we call the ‘spatial structure’ of the grid. This generalisation then finds application in two further themes, discussed next.

Theme 2: Numerical Grid Structure

A related sequence of tasks addresses the numerical structure of a grid. In the first of these, students are given two sets of square grid cards (1 × 1, 2 × 2, 3 × 3, 4 × 4 and 5 × 5). After exploring systematic ways in which they can be fitted next to or on top of each other, or in various formations or sequence, the teacher poses the questions, “Can you see a pattern? How many small squares are there on each card? What is the best way to find out?” Students then cut up a second set of grid cards into rows or columns, place the cut-outs on top of the first set of cards, and discuss the numbers of rows or columns and the number of small squares in each. After examining the resulting number pattern (1, 4, 9, 16, 25), the teacher removes all the grid cards and cut-outs and challenges the children to reproduce the visual pattern from memory, first on grid paper and then on plain paper. Discussion of similarities and differences between children’s drawings highlights the crucial fact that a square grid contains the same number of equally sized rows and columns. These ideas are further developed through a sequence of tasks focused on the pattern of squared numbers using grid cards.

In a follow-up task, students are given a 1 × 1 square and a 2 × 2 square and asked how many small squares fit on to the larger one. They are then given further 2 × 2 squares and asked to find the number of small squares in total, thus constructing the sequence 4, 8, 12, … . Finally, they are asked to generalise their findings. Heela had invented a perfectly good means of symbolising her results that closely resembles algebraic notation (Fig 3). In fact, she was treating the task as a functional relationship rather than a simple pattern continuation. Asked what she had learnt from the exercise, she said “I made a pattern so 1 big square is 4 little squares. So it’s 4 for each square. Every time you use the square it’s a four.” Further tasks showed that she had generalised the relationship to all sizes of square and, indeed, any type of rectangle.

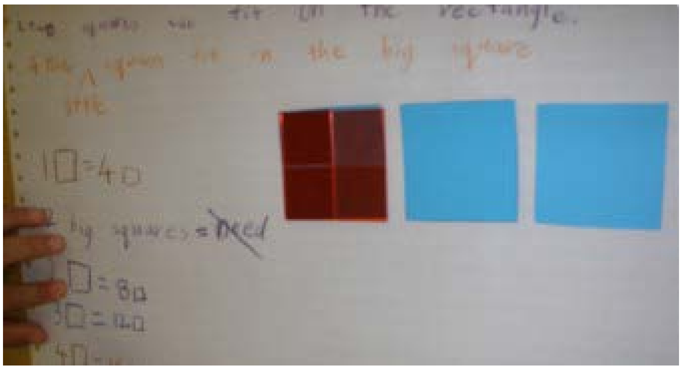


Figure 3. Heela’s response to Example Task 1.

Other tasks extend the basic (multiplicative) generalisation to rectangles, often in parallel to Theme 1. For example, in a task set late in Kindergarten, children are asked to relate the number of unit squares needed to cover a rectangle to the size of the unit.

Theme 3: Base Ten Numeration Structure

The generalisations learnt in Themes 1 and 2 can be applied to the structure of the base ten numeration structure, which is essentially multiplicative.

The first PASMAP activities in this theme explore the structure of 10-frames (5×2 grid filled with various numbers of dots) and use them to develop addition and subtraction strategies. The significance of structural understanding was already evident in the way that Heela and Lateh drew an empty 10-frame. Heela had no difficulty sketching 10-frames to represent any 1-digit number or in using them to add or subtract two such numbers. By contrast, Lateh could not draw any 10-frame or use them for addition or subtraction (see Figure 4).

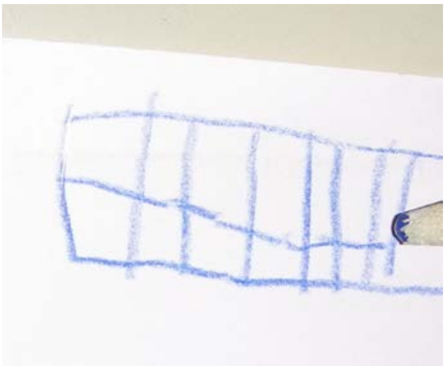
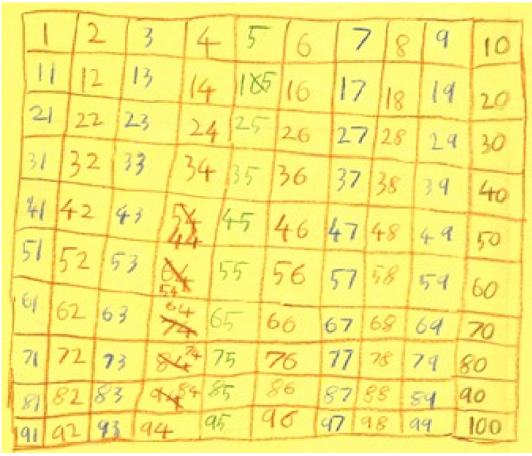


Figure 4. Lateh’s drawing of an empty 10-frame.

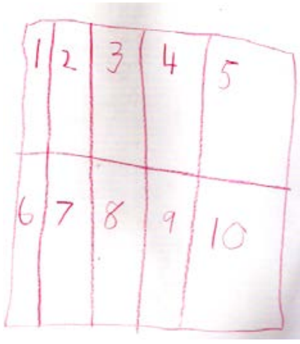
Another set of tasks focuses on the hundred chart. Building on earlier tasks involving on an empty numeral track, children gradually construct their own chart. They then identify various patterns in the chart, including multiples of 2, 5 and 10; cut the chart into rows and reassemble it; and write the numbers into an empty 10×10 grid.

Following these tasks, children are asked to draw a hundred chart from memory, firstly in an empty 10×10 grid and then freehand. The second task is much more difficult than the first. The teacher frequently prompts them to think about the spatial and numerical structure of the chart, for example “Does it have lots of rows?” and “How many boxes are there?”

During this process, children are encouraged to visualise and develop a plan of how to make their chart and to discuss it with the teacher.



Heela



Lateh

Figure 5. Two drawings of a hundred chart, drawn from memory.

Figure 5 shows how Heela and Lateh completed this task. Heela drew a good approximation to a square, drew the correct number of grid lines inside it, and then filled in multiples of 2 and 5 before completing the chart. Asked to explain her strategy, she said:

> We need 100 numbers, so 10 rows and 10 in each row. All numbers on the end in [each] column has to end with the same number. … It doesn't matter how far I go with the numbers, cause it's the tens column we are adding … one more ten each time. All the other numbers goes up by 1. I know how to count by all patterns!

Lateh's hundred chart only contained the numbers 1-10, and he claimed it had ten rows and columns. However, his drawing does show considerable structural development: He has learnt to draw a 5×2 grid without drawing separate squares, and he has reproduced the way that numbers are entered both into a 10-frame and a hundred chart. He has applied the structure of the 10-frame to the hundred chart without noticing the differences.

Discussion and Implications

The above examples show the special significance of the rectangular grid structure. Spatially, it is fundamental to early geometry, measurement, and graphical representation. Numerically, it arises whenever a unit is repeated—whether this unit be an object or a set of objects, a single shape or a composite shape, or a measurement unit. And algebraically, it lends itself to an early introduction to functional relationships and symbolisation. An early understanding of this structure is therefore vitally important to children's mathematical development.

Children's lack of understanding of structure can remain hidden when they only face tasks where the structure is provided for them. In all the examples we have shown, drawing from memory has been more revealing. Such tasks also challenge children to rehearse the patterns they have seen and apparently understood at a superficial level, and to use these patterns to gain a deeper understanding of the structure. In this sense, visual memory tasks are especially valuable, both for teachers and students.

The examples also show us once again that young children are capable of much more advanced mathematical thinking than was previously thought. The underlying principle, we would claim, is the way that PASMAP draws upon children's natural tendencies to look for patterns and then to explore how they are related. By encouraging students to continually seek patterns, to look for similarities and differences, and to form generalisations, they can learn about mathematics as relationships and can abstract and generalise, albeit at a simple level, from an early age.

We believe that the same structural approach could be applied throughout the teaching of mathematics and related areas of learning. We are currently extending our pedagogical approach in a 3-year longitudinal study of mathematics and science learning through novel experiences in data modelling and problem solving.[1] The study tracks three cohorts of students employed in the initial study (Mulligan et al., 2011a, 2011b) through to Grades 2, 3 and 4. In addition, two new cohorts of mathematically able students are being tracked from Kindergarten to Grade 2.

The implications of learning through a structural approach may require fundamental changes to the way that curriculum is conceptualised, structured, interpreted and implemented. The PASMAP approach promotes conceptual and connected knowledge and the development of teaching practices that focus on relational thinking and this may encourage the development of new pedagogical content knowledge (PCK). The Proficiency strands of the new Australian Curriculum–Mathematics (understanding, fluency, problem solving and reasoning) do support the development of mathematics as patterns, relationships and generalisations rather than disconnected concepts and skills (ACARA, 2012). PASMAP could play a major role in supporting the teaching of these proficiencies.

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