Evaluating Middle Years Students' Proportional Reasoning

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Proportional reasoning is a key aspect of numeracy that is not always developed naturally by students. Understanding the types of proportional reasoning that students apply to different problem types is a useful first step to identifying ways to support teachers and students to develop proportional reasoning in the classroom. This paper describes the development of a diagnostic instrument that aims to identify situations in which students can apply proportional reasoning and the types of reasoning they use.

Proportional reasoning is mathematical reasoning involving a sense of co-variation and multiple comparisons (Lesh, Post, & Behr, 1988). It requires multiplicative and relational thinking, which do not always develop naturally in students (Sowder et al., 1998). Lesh et al. (1988) described proportional reasoning as the cornerstone of higher level areas of mathematics, such as algebra, and the capstone of elementary concepts, such as arithmetic, number, and measurement. The importance of proportional reasoning goes beyond the mathematics classroom; its importance in other areas of study, for example, science, economics, and demographics has been widely recognized (Akatugba & Wallace, 2009; Boyer, Levine, & Huttenlocher, 2008). Perhaps more importantly, proportional reasoning is an essential aspect of everyday applications of numeracy. According to Ahl, Moore, and Dixon (1992), proportional reasoning is a "pervasive activity that transcends topical barriers in adult life" (p. 81) and yet, it has been estimated that more than half of the adult population are not proportional thinkers (Lamon, 1999).

Until recently, little was known about students' development or application of proportional reasoning (Lamon, 2005). Establishing the types and accuracy of reasoning used by students in situations of ratio and proportion can assist teachers in selecting teaching strategies and learning activities to target students' learning needs and strengthen their ability to reason proportionally (Misailidou & Williams, 2003). This paper describes the development of a diagnostic instrument designed to identify the types of reasoning used by students in a range of proportional and non-proportional situations and presents results from the pilot instrument to illustrate its use in diagnosing students' strategies and understanding.

Rationale

Several researchers have highlighted the importance of establishing children's proportional reasoning skills (e.g., Bright, Joyner, & Wallis, 2003; Misailidou & Williams, 2003; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). Methods that can be used to determine students' understanding include interviews, pen-and-paper tests (open-ended or multiple choice items), and concept mapping (Tüysüz, 2009). According to Wiggins and McTighe (1998), assessment must require students to explain or defend their

In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia). Singapore: MERGA.
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answer. While interviews provide a powerful method of establishing students' understanding, this is impractical in many situations, for example, in research situations where large numbers of students are involved. Tamir (1989) found that asking students for justifications when answering multiple choice questions is a sensitive and effective means of assessing students' understanding and argued that such an approach addresses some limitations commonly associated with traditional multiple-choice questions. To account for these arguments, Treagust (1995) recommended the use of two-tiered multiple-choice instruments as an alternative to individual interviews as a means of obtaining information about the reasoning of groups of students. The first tier of a two-tiered item consists of a multiple-choice question, with two to four choices. The second tier requires students to choose from four reasons to justify or explain their answer to the first tier question. To be considered correct, students must answer both levels of the question correctly. The data generated provide an insight into the common reasoning strategies employed by classes or year level cohorts.

For a number of years, science education researchers have designed and successfully used two-tiered multiple-choice instruments to diagnose students' understanding in a number of topics, as a means of informing pedagogical strategies (Chandrasegaran, Treagust, & Mocerino, 2008; Haslam & Treagust, 1987; Özmen, 2008; Tan & Treagust, 1999; Treagust, 1995, 2006; Tüysüz, 2009). O'Keefe and O'Donoghue (2011) used a two-tiered instrument based on Treagust's work to investigate the effectiveness of an intervention strategy in lower secondary mathematics classrooms; however, they did not provide examples of the items used. To date, such instruments have not been widely used for investigating students' understanding in mathematics.

Method

Instrument Design

The design of many two-tiered items is based on findings reported in research literature about students' common alternative conceptions, or the errors and difficulties they commonly encounter in a particular topic, concept, or reasoning situation. In the case of the instrument described here, research literature and findings from a previous study by members of the research team informed both the choice of problem type and the nature of the reasoning responses in the second tier of each item.

A detailed review of the literature in this area revealed varied descriptions of proportional reasoning problem types and the circumstances in which students typically use reasoning strategies correctly and incorrectly. A brief overview is provided here. Lamon (1993) identified four types of proportion problems: well-chunked or well-known measures (e.g., relationships that are commonly used rates, such as speed); part-part-whole (e.g., ratio problems in which two complementary parts are compared with each other or the whole); associated sets (rate situations in which the relationship between quantities is defined within the question); and stretchers and shrinkers (growth or scale problems). Lamon's categories formed the basis of the item types, however, Lesh et al. (1988) described several problem types that are neglected in textbooks, instruction, and research (e.g., problems involving between or within representation translations and missing value problems). Further, Van Dooren et al. (2005) suggested that often, students rely on proportional reasoning in circumstances that do not require it (e.g., constant, linear, and additive situations). Following a similar line of reasoning, Bright et al. (2003) stated that in order to assess students' ability to reason proportionally, it is important to provide situations in which students can correctly and incorrectly use multiplicative and additive thinking.

The items on the instrument were chosen to account for these arguments and suggestions and were of the following types:

- Non-proportional (constant or additive situations)
- Part-part-whole, missing value problems
- Scale (one- and two-dimensional)
- Rate (well-chunked measures, between representation translation)
- Relative - absolute (associated sets, within and between representation translation)
- Inverse proportion (well-chunked measures, associated sets)

Nine of the twelve items were based on items from the *Keeping it in Proportion* test *(KIIP),* used in a study by Dole, Clarke, Wright, and Hilton (2007). In that study, over 800 students (Years 5-9) completed the KIIP test, which employed short response questions that required students to show their working and provide written explanations of their calculated answers. These problems were re-worded to present a scenario and make a statement in the first tier, to which students responded *True* or *False*. To ensure that the instrument described here included items that targeted constant and additive non-proportional situations and two-dimensional scale, three additional items were based on problem types used by Bright et al. (2003) and Van Dooren et al. (2005).

The responses in the second tier of each item were based on the common strategies used by students in situations of proportion described in the literature. For example, multiplicative thinking is the critical factor for the comparison of quantities in proportion-related tasks. In contrast, additive reasoning involves considering the sums or differences in quantities (Bright et al., 2003). Students often find it difficult to discern which reasoning to use in a particular situation. Other difficulties associated with proportional situations and the errors commonly made by students that have been identified by researchers include

- an inability to discern when to use proportional reasoning and difficulty in identifying multiplicative or relative relationships (Van De Walle, Karp, & Bay-Williams, 2010);
- a tendency to approach proportional situations additively instead of multiplicatively (Cramer & Post, 1993; Lamon, 1993; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985);
- the use of multiplicative approaches unnecessarily, for example, when additive strategies should be used (Van Dooren, De Bock, & Verschaffel, 2010);
- inappropriate use of algorithms, such as cross multiplication (Nabors, 2003); and
- incorrect build-up/pattern building (Lamon, 1993; Misailidou & Williams, 2003).

In the instrument described here, second tier responses were used to identify students' correct and incorrect application of additive and multiplicative thinking, their recognition of absolute and relative situations, their ability to distinguish situations of proportion from non-proportion, and the strategies they used. In addition to the literature around students' difficulties in situations of proportion, the findings of the study by Dole et al. (2007) in which the KIIP test were used informed the wording of some options in the second tier of the items.

To illustrate the two-tier items, an example is provided. In the item, students were asked to compare two washing powders and to respond to the statement "Powder A is the better value". The students were provided with a graphic that portrayed the following information:

Powder A comes in 1 kg containers that cost $4 for 20 loads of washing.
Powder B comes in 1.5 kg containers that cost $6.50 for 30 loads of washing.

The second tier responses were

1. Washing powder A costs the least.
2. Washing powder B costs a little bit more but you get 10 more loads of washing.
3. The cost per load of washing is less.
4. Both washing powders are the same value.

This problem requires students to used relative thinking in a situation involving an associated set. Option 1 indicates that the student has used absolute thinking, comparing only price. Option 2 shows that the students have considered only the number of loads and not the price. Option 3 is the most accurate response and shows that students have used relative thinking. Option 4 suggests that students are considering the mass and number of loads but not the cost in their reasoning.

Administration of the Pilot Instrument

Prior to administering the instrument to the pilot group, the items were reviewed and revised by members of the research team (who have extensive mathematics teaching experience in primary and/or secondary schools) and a sample of teachers. The pilot instrument was administered to 140 Year 5 and 6 students (11-12 years old) in composite classes in two primary schools. Three classes of students from each school completed the instrument. Prior to administration of the instrument, the school principals were briefed on the purpose and administration procedure for the instrument. The teachers were provided with written instruction regarding the its administration.

The students' responses were coded and the percentage of students who responded to each alternative was calculated. The results of each class, school and the whole group were compared to determine whether the results were consistent or whether there were anomalies in the data. The results were similar for each group.

The combinations of responses to the first and second tiers allow identification of the students' reasoning in each item. Tan and Treagust (1999) suggested that when interpreting such data for the purpose of gaining an understanding of students' misconceptions, it is reasonable to consider those response combinations that exist for at least 10 per cent of students. The incorrect responses for which the percentage of students was greater than 10 per cent were further investigated to determine the types of reasoning used by the students.

Results

The data for the full sample of 140 students, shown in Table 1, show that the majority of students were able to answer Item 1 and just over half answered Item 7 correctly, however, for a number of items, a high percentage of students chose an incorrect combination.

Table 1
Percentage of Year 5 and 6 Students (n = 140) Selecting Each Response Combination

Item	Response							
	TA	FA	TB	FB	TC	FC	TD	FD
1	13.6	2.1	2.1	0.7	3.6	70.0*	3.6	0.7
2	5.0	5.0	8.6	45.7*	21.4	5.0	5.7	2.9
3	3.6	47.9	12.1	9.3	6.4	13.6	5.0*	2.9
4	1.4	9.3	3.6	32.9	14.3*	10.7	2.9	23.6
5	12.1	5.7	0.7	34.3*	1.4	6.4	2.1	33.6
6	15.7	1.4	4.3	42.9	27.9*	2.1	0.7	2.1
7	5.0	17.1	0.0	52.1*	6.4	8.6	5.0	6.4
8	2.9	23.6*	2.9	10.0	2.9	14.3	37.9	10.7
9	58.6	9.3	3.6	10.7	4.3	0.0	0.7	12.9*
10	4.3	2.1*	78.6	2.1	7.1	0.7	5.0	0.0
11	33.6	13.6	20.0*	7.1	4.3	10.0	0.7	7.1
12	0.7	21.4	4.3	13.6	19.3*	4.3	5.7	29.3

Notes. *indicates the correct response combination for the item. TA, FA, etc. denote first and second tier response combinations.

Examination of the combinations of incorrect responses selected by at least 10 per cent of the students allowed identification of the reasoning used in each situation. In some circumstances, the reasoning used indicated that the students had some level of qualitative understanding of the situation or relationship. The strategies used are shown in Table 2.

Table 2
Common Incorrect Strategies Used by Students

Strategy or error	Choice	%
Some qualitative understanding of relationship		
Understands need to compare part with whole	Item 3 (FC)	13.9
Qualitative understanding of speed-time relationship	Item 5 (FD)	33.6
Qualitative understanding of inverse relationship	Item 8 (D)	48.6
Some evidence of qualitative understanding of linear scale	Item 9 (FB)	10.7
Qualitative understanding of part-part-whole	Item 7 (FA)	17.1
Making absolute comparison instead of relative comparison		
Considering one value without considering the total	Item 3 (TB, FA)	60.0
Considering one value without considering a related value	Item 6 (TA)	15.7
	Item 12 (FA, FB)	35.0
Reading scale as absolute, not considering value of units	Item 9 (TA)	58.6

Using multiplicative strategies inappropriately		
Applying a multiplicative strategy to a constant situation	Item 1 (TA)	13.6
Applying a multiplicative strategy to an additive situation	Item 2 (TC)	21.4
Treating an inverse situation as if it involved direct proportion	Item 5 (TA)	12.1
Incorrect reasoning but recognizes inverse situation	Item 8 (FC)	14.3
No recognition of impact of changing two dimensions on area	Item 10 (TB)	78.6
Using additive strategies in proportional situation		
Increases two quantities by the same amount (direct)	Item 4 (FB, FD)	56.5
Increases two quantities by the same amount (inverse)	Item 8 (FB)	10.0
Erroneous calculation / no calculation		
Inaccurately uses multiplicative reasoning	Item 4 (FC)	10.7
Not calculating a value/estimating	Item 6 (FB)	42.9
Faulty understanding of visual representations		
Interprets graph accurately but chooses irrelevant response	Item 11 (A)	47.2
Misinterprets a distance-time graph	Item 11 (FC)	10.0
Misinterprets relative amounts in pictorial representation	Item 12 (FD)	29.3

Discussion and Conclusion

The results of the pilot instrument showed that it was useful for identifying the students' reasoning and the areas in which they applied incorrect reasoning strategies. The findings also aligned with the findings in the research regarding common errors and problems encountered by students in situations of proportion. As indicated by the data in Table 1, the students' success on the items varied. While many students answered some items correctly, the majority of students found particular items challenging, for example, Item 3 (requiring relative thinking) and Item 10 (requiring understanding of two-dimensional enlargement).

A major problem for students appears to be discriminating non-proportional from proportional situations. For example, for both non-proportional situations (Items 1 and 2), the majority of students who chose incorrectly used multiplicative thinking. Similarly, in proportional situations (Items 4 and 8), the majority of incorrect responses indicated the use of additive approaches. In addition to inappropriately using additive and multiplicative strategies, many students used absolute comparison in situations requiring relative thinking. For example, in several items, they only considered one variable when it was necessary to compare two parts or one part to the whole (Items 3, 6, 9).

It is important to note that in several items (Items 3, 5, 7, 8, 9), a number of students responded in ways that indicated a level of qualitative understanding of the relationship involved. This perhaps suggests that more targeted work with such students might assist them to develop a deeper understanding.

The data from the pilot instrument are for students in Years 5 and 6. As the final instrument is intended for use across Years 5 to 9, the items were designed to allow discrimination across year levels. This was found to be the case and may account for the low scores on some of the pilot items. For example, situations of inverse proportion (Items 5 & 8) are not usually a focus for students in primary years. Based on the data and feedback from the pilot, the original 12 items were retained for the final instrument. The items were arranged randomly so that no consecutive items were of the same problem type. Although some minor rewording was applied to make the meaning of the question clearer, the items and the second tier responses were retained.

The proportional reasoning two-tiered instrument described in this paper is a new approach to assessing students' proportional reasoning. The data generated from two-tiered instruments allows students' common errors or partial understanding to be identified, which informs researchers and teachers about the areas that teachers might target to enhance students' proportional reasoning. For example, those items that indicate students' use of absolute comparisons when they should make relative comparisons inform teachers that this is an area of weakness in their students. Such data may also be used to inform the design of teacher professional development to promote teachers' understanding of the different types of proportional reasoning and classroom strategies to address their students' needs.

The instrument described in this paper may be used as a pre-test and post-test to allow the researchers and teachers to track students' progress and to identify changes in their proportional reasoning to inform teachers' future curriculum planning and practice. In light of the importance of proportional reasoning development and the difficulties students encounter when reasoning proportionally, the various applications of this instrument have the potential to benefit teachers, curriculum planners, researchers and teacher educators.

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