Spatial Metaphors of the Number Line

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This paper examines spatial metaphors in the English language associated with the number line, in particular metaphors of direction and motion, and how these are manifested in actual spatial practices associated with number. It considers how these metaphors are culturally influenced, and how the influences of other cultures, such as Arabic, produce inconsistencies that can contribute to confusion in the classroom. It also considers how the metaphors of number vary in some other languages, and how this leads to both different spatial representations of numbers and more challenges in multilingual and multicultural classrooms. In particular, it pays attention to the implications of variety in these metaphors for the mathematics education of Indigenous students in Australia being taught by English speaking teachers.

Spatial metaphors associated with the number line are mixed and culturally specific, particularly in the assumption of a left to right directionality, which can contribute to confusion in the mathematics classroom. An analysis of these common metaphors can help teachers to identify potential confusion in their classrooms, particularly in multicultural and multilingual classrooms, such as those where Indigenous students in Australia are being taught by English speaking teachers. This paper reviews a diverse range of literature, raising critical questions for mathematics educators. In addition to literature from within the mathematics education field, it draws on cognitive science, especially research regarding the links between language and cognition, as well as historical, philosophical and anthropological perspectives. This discussion of the number line is focussed largely on positive numbers, as it is targeted towards the early years of primary mathematics education.

Two prevailing metaphors of numbers are used in mathematics education in the early years of school. Lakoff and Núñez (2000) articulate these metaphors as those used in daily life. The first is that numbers are collections of like objects. The second is that numbers are points on a line.

Both these metaphors include spatiality. The collection view of numbers has volume as a measure of magnitude – numbers are smaller or larger than each other. The number line metaphor introduces *location* and *direction*, but reduces magnitude to a single dimension. In terms of location, numbers have a fixed position on the line with respect to each other and the distance between them represents their magnitude. Although our visual representations of number lines are usually horizontal, we often use vertical terms to describe magnitude, with the larger numbers being higher and the smaller ones being lower (Watson, Partington, Gray, & Mack, 2006).

In terms of direction, the number line *starts* at zero or one and *goes to* infinity, often represented by an arrow head at the end of a written model of the number line (Figure 1). It thus has what Talmy (1996) describes as "fictive motion", in the sense that we say a road "goes" somewhere, when the road itself does not move and it is us who move along it. Fictive motion is extremely common in language. The fictive motion of the number line is of the type that Talmy labels an *orientation path*: "a continuous linear entity emerging from the front of some object and moving steadily away from it" (1996, p. 217). The origin of the number line is the zero point (or sometimes the point of one). This fictive motion thus introduces temporality as well as spatiality. Numbers can come *before* or *after* each other. A

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number that comes *before* another number is closer to the start of the number line than the other number, a number that comes *after* another is further from the start.

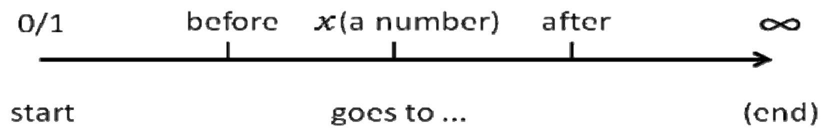

Figure 1. The number line.

Both *zero* and *one* are possible starting points for the number line, as there are cultures that do not have zero in their numbers, a relatively recent mathematical invention. Zero was developed as a solution to a need in written mathematics, specifically to a *spatial* need in written mathematics, as a place holder (Verran, 2001).

Direction of the Number Line

In Australia, number lines are usually drawn from left to right, the same direction as the English writing system. This convention is so ingrained that Lakoff and Núñez include left and right in their definition of the number line metaphor. For example they declare that "point *P* is to the right of point *Q*" maps onto "number *P'* is greater than number *Q'* " (2000: 279). This is the case in the English speaking world, but is not universal.

While Lakoff and Núñez (2000) recognise that ideas from different cultures find their way into mathematics, contributing to its evolution, they also claim that mathematics is independent of culture in the sense that "once mathematical ideas are established in a worldwide mathematical community, their consequences are the same for everyone, regardless of culture" (p. 356). Although Lakoff and Núñez admit diversity in mathematics, they focus more on mathematics as a singular thing and do not contextualise their definitions as culturally specific. Most of the time when most of us think about mathematics, we think of what Barton (2009) calls "near-universal conventional mathematics," or *Mathematics* as opposed to *mathematics* (Bishop, 1988). However, it is not that case that the consequences of mathematical ideas becoming conventionalised are "the same for everyone." Our conceptual metaphors are mediated by our languages are well as our cultural experiences. The directionality of the number line is a case in point.

The Mental Number Line

Many people have mental representations of number lines. Some of these are individual and idiosyncratic, others have culturally and historically determined characteristics. Galton (1881) surveyed and documented a variety of ways in which people visualised the layout of numbers. Around a hundred years later, Ernest (1986) conducted a similar survey. Galton claimed that around one man in 30 and one woman in 15 could visualise a stable number line. Ernest found 65 percent of his respondents claimed to be able to do so, a large increase.

The other significant difference between Ernest's findings and those of Galton was the increase in straight mental number lines. While Galton had found none, Ernest found 60 percent of his total respondents reported straight lines. Ernest reported his findings cautiously as he had only surveyed a small sample. In fact 90 percent of his respondents who did visualise a mental number line had a straight one. Ernest suggested the use of graduated rulers in mathematics instruction as a likely reason for both the increase in mental number lines and for their straightness. Another study of the idiosyncratic number visualizations of the type found by Galton found that the majority of the number forms reported were oriented left to right (Seron, Pesenti, Noel, Deloche, & Cornet, 1992).

Further evidence that number line features such as straightness and even its linearity are cultural constructs emerged from a study of number-space mappings among the Mundurukú, an Amazonian people who have very few number words. The study investigated how they chose to locate numbers on a line that was provided to them. The Mundurukú were found to represent numbers in a logarithmic fashion rather than a linear scale (Dehaene, Izard, Spelke, & Pica, 2008). Examination of some of the number forms in Galton (1881) and Seron et al. (1992) also suggest a logarithmic representation rather than linear. Dehaene et al. (2008) propose that the logarithmic representation of numbers is more innate than the linear, especially for estimation.

The SNARC Effect

The Spatial–Numerical Association of Response Codes or SNARC effect is described by Wood and Fisher (2008) thus: “We spontaneously associate numbers with space: we think of small numbers as being lower and to the left of us, and larger numbers as being further up and to the right of us” (p. 353). It was observed in experiments about reaction time in accessing information about number parity and magnitude from Arabic numerals (Dehaene, Bossini, & Giraux, 1993). French subjects responded faster to large numerals with their right hand and faster to small numerals with their left hand. This suggested mental number lines running from left to right. The SNARC effect appears to be a literacy effect related to the direction of the writing system. The study found a variation in performance on the task for Iranian subjects (Persian-French biliterate) that depended on the length of their exposure to the left to right writing system used in France. Another study found a reverse SNARC effect acting right to left among Arabic monoliterates (Zebian, 2005). A counter-explanation for the SNARC effect is that it emerges from finger-counting, with the right hand being used as the “counting tool” (Wood & Fischer, 2008). However, this does not explain the reverse SNARC effect. Nor is finger-counting a universal practice that Wood and Fischer claim (e.g. Butterworth, Reeve, & Reynolds, 2011).

Nevertheless, the SNARC effect provides evidence for the left to right direction of the mental number line among cultural groups who practice a left to right writing system. It also demonstrates that this has cognitive effects that extend to the motor system.

Discordance of Number Line with Direction of Writing

In light of the largely unconsciously engrained left to right directionality of the number line among people who write from left to right, it is interesting to examine the discordance of this direction with the direction of much arithmetic calculation.

The origins of this discordance stem from the European adoption of the Hindu numerals and place value system via the Arabic world. The Arabic writing system goes from right to left, and in Arabic numbers are written and read with their lowest power first. So “23” appears the same as in a European script, but it is written first “3” on the right, then “2” to the left of it, and it is read as “three and twenty” (Zebian, 2005).

When the Hindu number system was adopted into Europe by Fibonacci, he kept the order of writing as used by the Arabs. Because European languages such as Latin and English are written and read from left to right, the numbers began to be read from their highest power first (Devlin, 2011). So today in English we say “twenty three” and write the “2” first on the left, and the “3” next on the right.

However, standard methods of written calculation for addition, subtraction and multiplication are conducted from right to left. This is because we perform the operations on the lower powers first, the same as in the Arabic world. We thus need to learn and use a duality of direction of numbers, reading, writing, saying and thinking our numbers from left to right, but operating on them in columns from right to left. In the Arabic world, this duality of direction of numbers does not occur, since direction of reading and writing concords with the direction of calculation.

Cultural and Linguistic Variation in Spatial Representation of Number

The curved and twisted number lines found by Galton (1881) were two dimensional or even three dimensional rather than the one dimension of a straight line. How do the spatial representations of numbers differ for people who do not have or use number words, or who use numbers differently in their grammar? Like the Mundurukú language mentioned above, many Indigenous Australian languages traditionally have very few number words (but see Harris, 1982). Both Warlpiri, an Australian Central Desert language, and Iwaidja, spoken in Croker Island in the Northern Territory, have a "typical" Indigenous Australian number system: "one", "two" and "more than two", constructing "three" as "two and one" and "four" as "two and two". Anindilyakwa, spoken on Groote Island, has words for large numbers, unusually for an Australian language; however they are not much used (Stokes, 1982).

An investigation into the enumeration capabilities of Warlpiri and Anindilyakwa speaking children found that they were able to match and add concrete materials non-verbally but accurately despite having few number words. The researchers claimed that the Warlpiri and Anindilyakwa children "possess the same numerical concepts" as the English speaking comparison group (Butterworth, Reeve, Reynolds, & Lloyd, 2008, p. 13179). However, they also found that the Australian Indigenous language speaking children were more successful with spatial, pattern based approaches to the tasks than with enumeration, compared with English speaking children (Butterworth, Reeve, & Reynolds, 2011). It seems likely that these children do not have a mental number line with unnamed numbers located on it. It is more feasible that the Warlpiri and Anindilyakwa children have a collection concept of number. While the children in the study used two dimensional spatial strategies, these strategies were unique to each task rather than being stable for each individual. Investigations of mental number lines find that they are generally stable for individuals (Galton, 1881; Seron et al., 1992).

In some languages numbers are grammatically more like verbs than nouns (Barton, 2009). How might this affect the visualisation of a number line? Central to the number line metaphor is the idea that numbers are a "thing" that can be located. If numbers are perceived as actions, the mental visualisation of numbers could be very different. In Yoruba, spoken in Nigeria, numbers are nominalised verb phrases (Verran, 2001). Yoruba has a complex multi-base system that facilitates mental computation and traditionally was not written, nor did it use zero. Verran claims that the Yoruba conception of number is that they are "nested" groups: "each number is totally subsumed by its successor ... there is no sense of a linear stretching towards infinity" (p. 203). She argues that large part of the difference in the conception of numbers between English and Yoruba is because of the grammatical roles that they play in each language, noun-like and adjectival in English, and verb-al in Yoruba. Although the Yoruba do not represent numbers as points on in a line, the collection metaphor does not seem adequate either, since it is static, whereas the idea of numbers as verbs is dynamic.

The logarithmic representation of the Mundurukú is still a "line" in the common sense of the word. Other investigations would be required to explore other spatial representations of quantities. Investigations of non-linear spatial representations of numbers would need to be carefully designed so as not to impose researchers' preconceptions on the outcomes.

The Number Line in the Classroom

Although there is variety in the mental representations of number lines, the "standard" mental number line for people who speak a European language runs straight from left to right. In primary school mathematics classrooms in Australia, as with other places where European languages are used, students encounter physical models of the number line, most of which indeed run left to right. Common physical models of the number line used in primary schools include drawings on the board, strips of cardboard and of course, graduated rulers. As children internalise the left to right reading and writing direction, they also learn that this direction applies to the number line. Simple arithmetic problems are also written on a single line left to right.

Bove (1995) notes the confusion that children can experience when they move from experiences with a standard number line to trying to understand the place value system used in larger numbers: "multi-digit numerals increase in place value from right to left but numbers get larger in a horizontal number line from left to right" (p. 544). There is a lack of agreement between the spatial metaphors used in talking about number and writing them: "a number *higher* than six is actually written *beside and to the right* of it" (Watson, Partington, Gray, & Mack, 2006, p. 24) (emphasis in the original). Stacey, Helme and Steinle (2001) also identify the increase in place value from right to left as a source of confusion for students learning the value of decimals, sometimes leading them to see decimals and fractions as negative numbers as a result of mixing the directions of their spatial metaphors.

When children begin calculations of multi-digit numbers presented in columns, further confusion can occur. When children first encounter this type of calculation, the problems are usually levelled so that children are not required to carry digits across powers and they often attempt these written calculations from left to right. The left to right solution holds until they are required to carry digits, when the problem with the left to right approach becomes evident.

Bove (1995) suggests using a vertical number line commencing with zero at the bottom for the early introduction of numbers. This, she says "would not conflict with subsequent learning. In the vertical number line, numbers *increase* in value as the student moves *up* the line" (p. 544). To put it another way, the spatial and linguistic metaphors associated with the number line would match up with each other instead of contradicting each other. Teaching about the history of the system we use for writing numbers and its adoption from the right to left of Arabic would also help students to identify and understand the discordance.

As we have seen, the mixed spatial metaphors of the number line are a potential source of confusion for English-speaking children in an English-speaking classroom. These metaphors can be even more difficult for children from other linguistic backgrounds in an English-speaking mathematics classroom, such as Indigenous language speaking students in Australia.

Indigenous Students and the Number Line

As we saw above, teachers often mix their metaphors when talking about numbers quite unconsciously, assuming that children will be able to make the links between the various terms. This can be more difficult for students who are learning English as a Second or Additional Language (Watson, Partington, Gray, & Mack, 2006).

Indigenous language speaking students learning in English often have difficulty grasping the concepts of "before" and "after" in relation to numbers (Edmonds-Wathen, 2011; Graham, 1988). In part, this may be because while they have a collection metaphor for number, they are still developing the "numbers are points on a line" metaphor.

Another reason may be because the semantic scope of these terms in Australian Indigenous languages differs from their scope in English. For example, "it was found in one language that a word was being used for 'after' … [that] could be translated back into English as 'before', 'after', 'previously', 'following', etc., depending on the context" (Graham, 1988, p. 129).

In English, the number that comes "after" is "higher" or "greater", and the number that comes "before" is "lower" or "smaller". In Warlpiri, the concepts are also linked to height and size but in the inverse relation: The taller tree sprouted first and its smaller companions later, it is taller and older, it came "before"; the "bigger" brother came "first" and his "smaller" brother came "after" him, although the "bigger" brother's age in years is "greater". Thus in Warlpiri larger numbers a "larger" number can be seen to come "before" the "smaller" number (Mary Laughren, pers. comm., 11 March 2011).

Research into spatial concepts in Iwaidja had shown that similar to Warlpiri, the concepts of "first", "in front", "go before" are combined in one word, *wurdaka*, while "behind", "later" and "after" are combined in *warrwak.* Iwaidja also has a greater emphasis on *orientation* and implied or actual direction than *location* in spatial descriptions compared to English (Edmonds-Wathen, 2011). This orientation is very rarely described in terms of "left" and "right". We saw earlier that the type of fictive motion associated with the number line is what Talmy (1996) called an orientation path. Let us explore this metaphor a little more in terms of its implications for the number line when the emphasis is on orientation rather than location. In doing do, I am attempting a "mind-game" of the sort that Barton (2009) played with the mathematical implications of verbal expressions of shape in Navajo. I am not saying that this is exactly how speakers of these Australian Indigenous languages think.

The Path Oriented Approach

From a path oriented[1] approach to the number line, facing direction is always significant. Point on a line are never just scalar points, *they always have direction even if their apparent magnitude is zero*. From this perspective, any given number is facing away from its origin, travelling towards infinity. Larger number are thus before it, in front of it, while small numbers have been left behind, they are coming after it. The line is not going from left to right, or right to left, it is going *forward*.

This also correlates with the idea of increase with time, such that larger numbers began at zero, have been around for longer, getting bigger with time. The big mathematical problem with this type of thinking is that larger (cardinal) numbers have a lower ordinality than smaller numbers. "First" is always contextual, rather than associated with infinity.

The teacher could be more explicit about their use of fictive motion in talking about the number line. The temporal meaning of words such as *wurdaka* "first" and *warrwak* "later" could be emphasised in a journey, such that to reach six, you will have to pass five – it comes "first", it comes "before", and then if you keep going you will come to seven, it will come "later", it will come "after".

[1] Thanks to Mary Laughren for this phrasing.

Is there a way to map this approach onto the type of number line used in schools? Teachers in Indigenous classrooms are aware that they need to use concrete materials and physical processes in their mathematics teaching, such as number lines. Bove (1995) suggested uses vertical number lines to allay confusion over left and right practices when introducing place value. The vertical number line can also be lain flat in front of the body, stretching *away*.

Of course, I am not suggesting that simply using a number line pointing away from the body will be sufficient to allay the confusion that Indigenous students experience with the spatial metaphors of number in the classroom. What teachers need however, is an awareness of the metaphors that they are using and of how they may be being interpreted by their students.

Conclusion

We unconsciously absorb the dominant metaphors associated with important concepts from our culture and language. The number line metaphor, which materialises as physical models of number lines within and without classrooms, has spatial associations. The standard number line metaphor in English has both orientation and direction, but location at fixed points is regarded as more important than the fictive journey along it. While possibly we begin with a logarithmic representation of this journey, we learn to divide it into equal sections and name each point as a number. These numbers themselves have no direction. There are different ways of conceptualising spatially quantity and number in different cultures and languages. For some, the number as a collection metaphor may be more highly developed than that of a points on a line. Concepts of ordinality may vary, and for some such as the Warlpiri a “bigger” number may come “before” a “smaller”. Resolving metaphorical contradictions in the mathematical classrooms is no easy matter, yet even in English alone, we learn to mix our mathematical metaphors. For the English speaking teacher, an awareness of these common metaphors in English, and how they differ from those in other languages that might be spoken by their students, can assist them in explicit teaching about how one metaphor maps onto another. An appreciation of the Arabic impact on the directionality of how we write numbers might help us clarify to our students that we use both left to right and right to left in our number manipulations. This is even more important when students come from cultures that commonly do not use the right/left distinction and which focus more on direction and movement than on location.

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